

# Spinor calculus for $q$ -deformed quantum spaces II

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## Abstract

This is the second part of an article about  $q$ -deformed analogs of spinor calculus. The considerations refer to quantum spaces of physical interest, i.e.  $q$ -deformed Euclidean space in three or four dimensions as well as  $q$ -deformed Minkowski space. The Clifford algebras corresponding to these quantum spaces are treated. Especially, their commutation relations and their Hopf structures are written down. Bases of the four-dimensional Clifford algebras are constructed and their properties are discussed. Matrix representations of the Clifford algebras lead to  $q$ -deformed  $\gamma$ -matrices for the four-dimensional quantum spaces. Moreover,  $q$ -analogs of the four-dimensional spin matrices are presented. A very complete set of trace relations and rearrangement formulae concerning spin and  $\gamma$ -matrices is given. Dirac spinors together with their bilinear covariants are defined. Their behavior under  $q$ -deformed Lorentz transformation is discussed in detail.

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# 1 Introduction

Spinors play a very important role in physics. As is well-known, they are indispensable for describing particles with spin  $1/2$ . Moreover, spinors are an essential ingredient of supersymmetry. In part I of this article (see Ref. [1]) we started studying a  $q$ -deformed version of spinor calculus. The motivation for this undertaking stems from the observation that  $q$ -deformation [2–15] should provide a very promising approach for discretizing space-time [16–20] (for other deformations of space-time see Refs. [21–27]). Remember that formulating quantum field theories on a lattice-like space-time structure is a very appealing idea, since it should lead to an effective method for regularizing quantum field theories [28]. Due to its attractiveness, one can find a number of attempts to attack the problem of discretizing space-time in the former literature (see for example Refs. [29–34]).

The construction of a  $q$ -deformed quantum field theory requires a very deep and thoughtful understanding of the subject. Thus, we gave in part I of the article a review of the foundations of the quantum group  $SU_q(2)$  [35–38]. Additionally, we considered their two-dimensional co-representations, which are known as symmetrized and antisymmetrized Manin plane (or quantum plane). The coordinates of the Manin plane can be viewed as  $q$ -deformed spinors [39–43]. Furthermore, we described the construction of higher dimensional quantum spaces out of quantum planes. In doing so, we restricted attention to  $q$ -deformed Minkowski space and  $q$ -deformed Euclidean space with three- or four-dimensions. These results enabled us to introduce  $q$ -analogs of the well-known Pauli matrices. Although our reasonings were in remarkable analogy to the undeformed case we had to take care of a very delicate structure. In the  $q$ -deformed setting there are different types of Pauli matrices to distinguish, so we have carefully to decide which of these Pauli matrices applies in a certain situation. For this reason, part I of the article provides the reader with a detailed and rather complete discussion of the properties of these Pauli matrices. In addition to this, we introduced  $q$ -analogs of spin matrices and gave a comprehensive list of calculational rules and rearrangement formulae that could prove useful in physical applications. Especially, we presented  $q$ -analogs of the very important Weyl Spinor Fierz identities.

In part II we now continue our reasonings about  $q$ -deformed spinor calculus. However, stress is taken on a presentation that reveals the foundations of a  $q$ -deformed Dirac formalism. Thus, we use our results from part I to

define  $q$ -analogs of Dirac matrices and spin matrices. In this article the Dirac matrices ( $\gamma$ -matrices) establish representations of the Clifford algebras to four-dimensional quantum spaces, i.e.  $q$ -deformed Euclidean space in four dimensions and  $q$ -deformed Minkowski space. In Sec. 2 we write down the commutation relations and Hopf structures of each Clifford algebra under consideration, where for the sake of completeness we also deal with the Clifford algebra of three-dimensional  $q$ -deformed Euclidean space.

In analogy to the undeformed case a basis of a  $q$ -deformed Clifford algebra can be built up from antisymmetrized products of Clifford generators. We also define a quantum trace which together with the  $\gamma$ -matrices as representations of the Clifford generators enables us to calculate a left- as well as a right-dual basis. This task will be done in Sec. 3.1.1 and Sec. 3.2.1. In the undeformed case the left-dual basis is identical to the right-dual one, but in the  $q$ -deformed case it is not. The reason for this lies in the fact that for the quantum spaces under consideration the metric is not symmetrical. Finally, we will see that each dual basis can be used to write down some sort of completeness relation.

The main part of the paper [cf. Sec. 3.1.2, Sec. 3.1.3, Sec. 3.2.2, and Sec. 3.2.3] is devoted to trace relations and rearrangement formulae concerning  $q$ -deformed  $\gamma$ -matrices and spin matrices. For the undeformed counterparts of these identities we refer the reader to Refs. [44–49], for example. Our reasonings apply to both  $q$ -deformed Minkowski space and four-dimensional  $q$ -deformed Euclidean space. Nevertheless, there are some remarkable differences between these spaces. Due to the various types of Pauli matrices of  $q$ -deformed Minkowski space the definitions of  $\gamma$ -matrices and spin matrices are not unique for that space. Thus, the discussion of trace relations and rearrangement formulae is case sensitive in  $q$ -deformed Minkowski space.

Let us also note that in complete analogy to part I we derived many identities by first proposing reasonable ansätze. Then we inserted the expressions for  $\gamma$ -matrices and spin matrices into each ansatz and obtained a system of equations for its unknown coefficients. The corresponding solutions were found by means of a computer algebra system like *Mathematica* [50].

Section 4 covers the subject of  $q$ -deformed Dirac spinors and their bilinear covariants. Working in a kind of Weyl representation Dirac spinors together with their Hermitian and Dirac conjugates are introduced in very much the same way as is done for the undeformed case, i.e. we simply write a 4-spinor as two 2-spinors. The corresponding bilinear covariants are of the same form as in the undeformed case. We discuss the conjugation properties of

these covariants. Furthermore, we carefully examine the behavior of the different types of 4-spinors under  $q$ -deformed Lorentz transformations. From these results we find that our bilinear covariants again transform as scalar, pseudoscalar, vector, pseudovector, and antisymmetric tensor. In summary, our considerations reveal a very remarkable similarity to the undeformed situation (see, for example, Ref. [51]).

We end our reasonings with a short conclusion in Sec. 5. In App. A and App. B we collected some expressions referring to four-dimensional  $q$ -deformed Euclidean space and  $q$ -deformed Minkowski space, respectively. Especially, we listed matrix representations of the various Clifford bases and expressions for spin matrices.

Last but not least let us make some notational remarks. Throughout the article we use the shortcuts  $\lambda \equiv q - q^{-1}$  and  $\lambda_+ \equiv q + q^{-1}$ . Sometimes an  $n$ -dimensional identity matrix is written as  $\mathbb{1}$ . The antisymmetric  $q$ -numbers are defined by  $[[n]]_{q^a} \equiv \frac{1-q^{an}}{1-q^a}$ , where we assume  $q > 1$  and  $a \in \mathbb{R}$ .

## 2 $q$ -Deformed Clifford algebras

This section is devoted to the Clifford algebras that correspond to quantum spaces we are interested for physical reasons, i.e. three- and four-dimensional  $q$ -deformed Euclidean space as well as  $q$ -deformed Minkowski space. For each Clifford algebra we give a short review of its defining relations and discuss its Hopf structures. In the subsequent sections  $q$ -analogs of  $\gamma$ -matrices are introduced as four-dimensional representations of Clifford generators.

### 2.1 Clifford algebra of three-dimensional $q$ -deformed Euclidean space

The Clifford algebra of three-dimensional  $q$ -deformed Euclidean space is spanned by three generators  $\xi^k$ ,  $k \in \{+, 3, -\}$ . Their commutation relations are determined by two conditions [8]. The first one is

$$(P_S)^{ij} \xi^k \xi^l = 0, \quad (1)$$

where  $P_S$  denotes the symmetrizer of the three-dimensional  $q$ -deformed Euclidean space. As a second claim we have

$$(P_0)^{ij} \xi^k \xi^l = c g^{ij} \mathbb{1}, \quad (2)$$

where  $P_0$  stands for the projector proportional to the quantum metric  $g^{ij}$  of three-dimensional  $q$ -deformed Euclidean space. Notice that the constant  $c \in \mathbb{R}$  is not fixed yet. If not stated otherwise, there is always summation over repeated indices understood.

Using the projector decomposition of the  $\hat{R}$ -matrix of three-dimensional  $q$ -deformed Euclidean space [15, 52], i.e.

$$\hat{R} = P_+ - q^{-4}P_- + q^{-6}P_0, \quad (3)$$

the Clifford algebra can be written in the following compact form:

$$\xi^i \xi^j = -q^4 \hat{R}^{ij}_{kl} \xi^k \xi^l + c q^{-1} \lambda_+ g^{ij} \mathbb{1}. \quad (4)$$

However, starting from the projector decomposition of  $\hat{R}^{-1}$  would yield

$$\xi^i \xi^j = -q^{-4} (\hat{R}^{-1})^{ij}_{kl} \xi^k \xi^l + c q \lambda_+ g^{ij} \mathbb{1}. \quad (5)$$

More explicitly, the Clifford algebra in (4) reads as

$$\begin{aligned} \xi^+ \xi^+ &= \xi^- \xi^- = 0, \\ \xi^3 \xi^\pm &= -q^{\mp 2} \xi^\pm \xi^3, \\ \xi^- \xi^+ &= -\xi^+ \xi^- - c \lambda_+ \mathbb{1}, \\ \xi^3 \xi^3 &= \lambda \xi^+ \xi^- + c q^2 \mathbb{1}. \end{aligned} \quad (6)$$

Exploiting (5) would lead us to the same commutation relations.

In part I of the paper we showed that the Pauli matrices  $\sigma^i$ ,  $i \in \{+, 3, -\}$ , for three-dimensional  $q$ -deformed Euclidean space also fulfill a Clifford algebra. It becomes identical to (6) if  $c$  takes on the value  $q^2 \lambda_+^{-1}$ .

It is possible to endow the Clifford algebra (6) with two Hopf structures. Their explicit form on the generators of the Clifford algebra can easily be found from the Hopf structures for the quantum space coordinates  $X^i$  (as they are given in Refs. [52, 53], for example) by substituting the Clifford generators  $\xi^i$  for the quantum space coordinates. Thus, the corresponding coproducts, antipodes, and counits for the Clifford generators become

$$\begin{aligned} \Delta(\xi^i) &= \xi^i \otimes 1 + \mathcal{L}^i_j \otimes \xi^j \\ \bar{\Delta}(\xi^i) &= \xi^i \otimes 1 + \bar{\mathcal{L}}^i_j \otimes \xi^j, \\ S(\xi^i) &= -S(\mathcal{L})^i_j \xi^j, \end{aligned}$$

$$\begin{aligned}\bar{S}(\xi^i) &= -S(\bar{\mathcal{L}})^i_j \xi^j, \\ \epsilon(\xi^i) &= \bar{\epsilon}(\xi^i) = 0,\end{aligned}\tag{7}$$

where  $\mathcal{L}$  and  $\bar{\mathcal{L}}$  stand for the  $L$ -matrix and its conjugate, respectively. Let us recall that the  $L$ -matrix and its conjugate can be viewed as realizations of the  $SU_q(2)$ -quantum group generators within the quantum algebra  $U_q(su(2))$ .

However, there is one subtlety we have to take care of, since we cannot assign the usual group-like coproduct to  $\mathbb{1}$ . Instead we have

$$\Delta(\mathbb{1}) = \mathbb{1} \otimes 1 + \Lambda^{-1} \otimes \mathbb{1} - q^3 c^{-1} \lambda g_{ij} \xi^i \mathcal{L}^j_k \otimes \xi^k,\tag{8}$$

and

$$\bar{\Delta}(\mathbb{1}) = \mathbb{1} \otimes 1 + \Lambda \otimes \mathbb{1} - q^{-3} c^{-1} \lambda g_{ij} \xi^i \bar{\mathcal{L}}^j_k \otimes \xi^k.\tag{9}$$

Inserting the explicit form for the entries of the  $L$ -matrices (see, for example, Ref. [52]) we get

$$\begin{aligned}-q^3 c^{-1} \lambda g_{ij} \xi^i \mathcal{L}^j_k \otimes \xi^k &= -c^{-1} \lambda \Lambda^{-1/2} [q^2 \xi^+(\tau^3)^{-1/2} \otimes \xi^- \\ &\quad - q \xi^3 \otimes \xi^3 - q \lambda \lambda_+ \xi^3 L^+ \otimes \xi^- \\ &\quad + \xi^-(\tau^3)^{1/2} \otimes \xi^+ + q \lambda \lambda_+ \xi^-(\tau^3)^{1/2} L^+ \otimes \xi^3 \\ &\quad + q^2 \lambda^2 \lambda_+ \xi^-(\tau^3)^{1/2} (L^+)^2 \otimes \xi^-],\end{aligned}\tag{10}$$

and

$$\begin{aligned}-q^{-3} c^{-1} \lambda g_{ij} \xi^i \bar{\mathcal{L}}^j_k \otimes \xi^k &= c^{-1} \lambda \Lambda^{1/2} [q^{-2} \xi^-(\tau^3)^{-1/2} \otimes \xi^+ \\ &\quad - q^{-1} \xi^3 \otimes \xi^3 - q^{-1} \lambda \lambda_+ \xi^3 L^- \otimes \xi^+ \\ &\quad + \xi^+(\tau^3)^{1/2} \otimes \xi^- + q^{-1} \lambda \lambda_+ \xi^+(\tau^3)^{1/2} L^- \otimes \xi^3 \\ &\quad + q^{-2} \lambda^2 \lambda_+ \xi^+(\tau^3)^{1/2} (L^-)^2 \otimes \xi^+],\end{aligned}\tag{11}$$

where  $L^+$ ,  $L^-$ , and  $\tau^3$  are generators of  $U_q(su(2))$ , while  $\Lambda$  stands for a unitary scaling operator subject to

$$\Lambda \mathbb{1} = q^{-8} \mathbb{1} \Lambda, \quad \Lambda^{1/2} \xi^k = -q^{-2} \xi^k \Lambda^{1/2}, \quad k \in \{+, 3, -\}.\tag{12}$$

Notice that these scaling properties are consistent with the requirement that the scaling operator respects the relations in (4) and (5). Finally, it should be mentioned that the above expressions for the coproducts  $\Delta(\mathbb{1})$  and  $\bar{\Delta}(\mathbb{1})$  are consistent with

$$\begin{aligned}S(\mathbb{1}) &= -q^6 \Lambda \mathbb{1}, \quad \bar{S}(\mathbb{1}) = -q^{-6} \Lambda^{-1} \mathbb{1}, \\ \epsilon(\mathbb{1}) &= \bar{\epsilon}(\mathbb{1}) = 0.\end{aligned}\tag{13}$$

## 2.2 Clifford algebra of four-dimensional $q$ -deformed Euclidean space

The Clifford algebra for the four-dimensional  $q$ -deformed Euclidean space is constructed in very much the same way as was done for the three-dimensional  $q$ -deformed Euclidean space. Again, we start from the requirements

$$(P_S)^{ij}{}_{kl} \xi^k \xi^l = 0, \quad (P_0)^{ij}{}_{kl} \xi^k \xi^l = c g^{ij} \mathbb{1}, \quad (14)$$

but now we have to use the projectors of the four-dimensional  $q$ -deformed Euclidean space [38]. From (14) together with the projector decomposition of the  $\hat{R}$ -matrix of the quantum group  $SO_q(4)$  (see, for example, Ref. [52]) we can derive as defining relations of the Clifford algebra

$$\xi^i \xi^j = -q \hat{R}_{kl}^{ij} \xi^k \xi^l - q^{-1} c \lambda_+ g^{ij} \mathbb{1}. \quad (15)$$

Alternatively, we get

$$\xi^i \xi^j = -q^{-1} (\hat{R}^{-1})_{kl}^{ij} \xi^k \xi^l - q c \lambda_+ g^{ij} \mathbb{1}, \quad (16)$$

if we instead work with the projector decomposition of  $\hat{R}^{-1}$ . Both (15) and (16) imply the following commutation relations for the Clifford generators  $\xi^k$ ,  $k = 1, \dots, 4$ :

$$\begin{aligned} \xi^k \xi^k &= 0, & k &= 1, \dots, 4, \\ \xi^j \xi^1 &= -q \xi^1 \xi^j, \\ \xi^4 \xi^j &= -q \xi^j \xi^4, & j &= 2, 3, \\ \xi^4 \xi^1 &= -\xi^1 \xi^4 - c \lambda_+ \mathbb{1}, \\ \xi^3 \xi^2 &= -\xi^2 \xi^3 + \lambda \xi^1 \xi^4 - q^{-1} c \lambda_+ \mathbb{1}. \end{aligned} \quad (17)$$

In Sec. 3.1 we will calculate  $\gamma$ -matrices for the four-dimensional  $q$ -deformed Euclidean space. These  $\gamma$ -matrices will obey the relations in (17) if we choose  $c = -\lambda_+^{-1}$ .

In analogy to the three-dimensional case there exist two Hopf structures for the Clifford algebra of four-dimensional  $q$ -deformed Euclidean space. Once again, these Hopf structures are of the same form as those for the coordinates of four-dimensional  $q$ -deformed Euclidean space [cf. the expressions in (7) with  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  now standing for the  $L$ -matrices of  $U_q(so(4))$ ].



Next, we come to the coproducts of  $\mathbb{1}$  for which we have to set

$$\begin{aligned}\Delta(\mathbb{1}) &= \mathbb{1} \otimes 1 + \Lambda^{-1} \otimes \mathbb{1} + qc^{-1}\lambda g_{ij} \xi^i \mathcal{L}^j_k \otimes \xi^k, \\ \bar{\Delta}(\mathbb{1}) &= \mathbb{1} \otimes 1 + \Lambda \otimes \mathbb{1} + q^{-1}c^{-1}\lambda g_{ij} \xi^i \bar{\mathcal{L}}^j_k \otimes \xi^k.\end{aligned}\quad (18)$$

Written out explicitly the non-classical contribution to the coproduct  $\Delta(\mathbb{1})$  reads as

$$\begin{aligned}qc^{-1}\lambda g_{ij} \xi^i \mathcal{L}^j_k \otimes \xi^k &= -q^{-1}c^{-1}\lambda \Lambda^{-1/2} \left[ -q^2\lambda^2 \xi^1 K_1^{1/2} K_2^{1/2} L_1^+ L_2^+ \otimes \xi^1 \right. \\ &\quad - q\lambda \xi^1 K_1^{-1/2} K_2^{1/2} L_2^+ \otimes \xi^2 - q\lambda \xi^1 K_1^{1/2} K_2^{-1/2} L_1^+ \otimes \xi^3 \\ &\quad + \xi^1 K_1^{-1/2} K_2^{-1/2} L_1^+ L_2^+ \otimes \xi^4 + q^2\lambda \xi^2 K_1^{1/2} K_2^{1/2} L_2^+ \otimes \xi^1 \\ &\quad - q\xi^2 K_1^{1/2} K_2^{-1/2} L_1^+ L_2^+ \otimes \xi^3 - q^2\lambda \xi^3 K_1^{1/2} K_2^{1/2} L_1^+ \otimes \xi^1 \\ &\quad \left. - q\xi^3 K_1^{-1/2} K_2^{1/2} \otimes \xi^2 + q^2\xi^4 K_1^{1/2} K_2^{1/2} \otimes \xi^1 \right].\end{aligned}\quad (19)$$

The corresponding expression for the conjugated coproduct can be obtained from (19) by applying the substitutions

$$\xi^1 \leftrightarrow \xi^4, \quad \xi^2 \leftrightarrow \xi^3, \quad L_1^+ \rightarrow L_1^-, \quad L_2^+ \rightarrow L_2^-, \quad q \rightarrow q^{-1}. \quad (20)$$

Notice that the elements  $L_i^\pm$ ,  $K_i$ ,  $i = 1, 2$ , generate the quantum algebra  $U_q(so(4))$  and the scaling operator  $\Lambda$  obeys the commutation relations

$$\Lambda \mathbb{1} = q^{-4} \mathbb{1} \Lambda, \quad \Lambda^{1/2} \xi^k = -q^{-1} \xi^k \Lambda^{1/2}, \quad k = 1, \dots, 4. \quad (21)$$

For the antipodes and counits of  $\mathbb{1}$  we have

$$\begin{aligned}S(\mathbb{1}) &= -q^4 \Lambda \mathbb{1}, \quad \bar{S}(\mathbb{1}) = -q^{-4} \Lambda^{-1} \mathbb{1}, \\ \epsilon(\mathbb{1}) &= \bar{\epsilon}(\mathbb{1}) = 0.\end{aligned}\quad (22)$$

### 2.3 Clifford algebra of $q$ -deformed Minkowski space

The Clifford algebra to  $q$ -deformed Minkowski space is again determined by the requirements

$$(P_S)^{\mu\nu}{}_{\rho\sigma} \xi^\rho \xi^\sigma = 0, \quad (P_0)^{\mu\nu}{}_{\rho\sigma} \xi^\rho \xi^\sigma = c\eta^{\mu\nu} \mathbb{1}, \quad (23)$$

where the projectors as well as the quantum metric  $\eta^{\mu\nu}$  now refer to  $q$ -deformed Minkowski space. Using the projector decomposition of the  $\hat{R}$ -matrix of  $q$ -deformed Minkowski space (see Ref. [15], for example) the above conditions can be combined to give the following equivalent relations:

$$\xi^\mu \xi^\nu = -\hat{R}^{\mu\nu}{}_{\rho\sigma} \xi^\rho \xi^\sigma + cq\lambda_+ \eta^{\mu\nu} \mathbb{1}, \quad (24)$$

$$\xi^\mu \xi^\nu = -(\hat{R}^{-1})^{\mu\nu}{}_{\rho\sigma} \xi^\rho \xi^\sigma + cq^{-1} \lambda_+ \eta^{\mu\nu} \mathbb{1}. \quad (25)$$

From (24) as well as (25) we found as independent commutation relations

$$\begin{aligned} \xi^A \xi^A &= 0, \quad A \in \{+, -\}, \\ \xi^3 \xi^\pm &= -q^{\mp 2} \xi^\pm \xi^3, \\ \xi^3 \xi^3 &= \lambda \xi^+ \xi^- + cq^2 \mathbb{1}, \\ \xi^0 \xi^\pm &= -\xi^\pm \xi^0 \pm q^{\mp 1} \lambda \xi^\pm \xi^3, \\ \xi^0 \xi^3 &= -\xi^3 \xi^0 + \lambda \xi^+ \xi^- + qc \lambda \mathbb{1}, \\ \xi^0 \xi^0 &= -c \mathbb{1}, \\ \xi^- \xi^+ &= -\xi^+ \xi^- - c \lambda_+ \mathbb{1}. \end{aligned} \quad (26)$$

It should also be mentioned that (26) contains the relations of the Clifford algebra to three-dimensional  $q$ -deformed Euclidean space. Indeed, dropping all relations with  $\xi^0$  leaves us with the relations in (6).

In Sec. 3.2 we are going to introduce  $\gamma$ -matrices for  $q$ -deformed Minkowski space. The relations obtained from (26) for the choice  $c = -\lambda_+^{-1}$  are fulfilled by these  $\gamma$ -matrices.

In dealing with monomials of the  $\xi^\mu$  it is sometimes convenient to bring them into a normal ordered form. In this respect, the following formulae derived from the relations in (26) could prove useful:

$$\begin{aligned} (\xi^0)^{2n} &= (-1)^n c^n \mathbb{1}, \quad (\xi^0)^{2n-1} = (-1)^{n-1} c^{n-1} \xi^0, \\ (\xi^3)^{2n} &= q^{2n} c^n \mathbb{1} + q^{-2(n-1)} c^{n-1} \lambda [[n]]_{q^4} \xi^+ \xi^-, \\ (\xi^3)^{2n-1} &= q^{2n} c^n \xi^3 - q^{-2n} c^{n-1} \lambda [[n]]_{q^4} \xi^+ \xi^3 \xi^-, \\ (\xi^0)^{2n-1} (\xi^3)^{2n-1} &= (-1)^n (cq)^{2n} [-q \lambda \mathbb{1} + c^{-1} \xi^3 \xi^0 \\ &\quad - c^{-1} \lambda \xi^+ \xi^- + c^{-2} q^{-4n} \lambda [[n]]_{q^4} \xi^+ \xi^3 \xi^0 \xi^-]. \end{aligned} \quad (27)$$

To make the Clifford algebra of  $q$ -deformed Minkowski space into a Hopf algebra we proceed in very much the same way as was done for the Euclidean spaces. Again, coproduct, antipode, and counit for the generators  $\xi^\mu$  are of the same form as for the quantum space coordinates  $X^\mu$ . For  $\mathbb{1}$  we have to take

$$\begin{aligned} \Delta(\mathbb{1}) &= \mathbb{1} \otimes \mathbb{1} + \Lambda^{-1} \otimes \mathbb{1} + q^{-2} c^{-1} \lambda \lambda_+^{-1} \eta_{\mu\nu} \xi^\mu \mathcal{L}^\nu{}_\kappa \otimes \xi^\kappa, \\ \bar{\Delta}(\mathbb{1}) &= \mathbb{1} \otimes \mathbb{1} + \Lambda \otimes \mathbb{1} - q^2 c^{-1} \lambda \lambda_+^{-1} \eta_{\mu\nu} \xi^\mu \bar{\mathcal{L}}^\nu{}_\kappa \otimes \xi^\kappa, \end{aligned} \quad (28)$$

and

$$S(\mathbb{1}) = -q^{-4}\Lambda\mathbb{1}, \quad \bar{S}(\mathbb{1}) = -q^4\Lambda^{-1}\mathbb{1}. \quad (29)$$

The non-classical contributions in (28) can be expressed in terms of the generators of the  $q$ -deformed Lorentz algebra [12, 14], i.e.

$$\begin{aligned} & q^{-2}c^{-1}\lambda\lambda_+^{-1}\eta_{\mu\nu}\xi^\mu\mathcal{L}^\nu{}_\kappa\otimes\xi^\kappa = \\ & = -q^{-5/2}c^{-1}\lambda\lambda_+^{-1}\Lambda^{-1/2}\left[q^{5/2}\lambda^2\xi^+(\tau^3)^{-1/2}T^-S^1\otimes\xi^+ \right. \\ & \quad - \lambda\lambda_+^{-1/2}\xi^+T^-\tau^1\otimes\xi^3 + q^3\lambda\lambda_+^{-1/2}\xi^+S^1\otimes\xi^3 \\ & \quad + \lambda\lambda_+^{-1/2}\xi^+T^-\tau^1\otimes\xi^0 + q\lambda\lambda_+^{-1/2}\xi^+S^1\otimes\xi^0 \\ & \quad - q^{5/2}\xi^+(\tau^3)^{1/2}\tau^1\otimes\xi^- - q\lambda\lambda_+^{-1/2}\xi^3(\tau^3)^{-1/2}S^1\otimes\xi^+ \\ & \quad + q^2\lambda\lambda_+^{-1/2}\xi^3(\tau^3)^{-1/2}T^-\sigma^2\otimes\xi^+ + q^{1/2}\lambda_+^{-1}\xi^3\tau^1\otimes\xi^3 \\ & \quad + q^{5/2}\lambda_+^{-1}\xi^3\sigma^2\otimes\xi^3 - q^{3/2}\lambda^2\lambda_+^{-1}\xi^3T^-T^2\otimes\xi^3 \\ & \quad - q^{1/2}\lambda_+^{-1}\xi^3\tau^1\otimes\xi^0 + q^{1/2}\lambda_+^{-1}\xi^3\sigma^2\otimes\xi^0 \\ & \quad + q^{3/2}\lambda^2\lambda_+^{-1}\xi^3T^-T^2\otimes\xi^0 - q^4\lambda\lambda_+^{-1/2}\xi^3(\tau^3)^{1/2}T^2\otimes\xi^- \\ & \quad - q^2\lambda\lambda_+^{-1/2}\xi^0(\tau^3)^{-1/2}T^-\sigma^2\otimes\xi^+ - q^3\lambda_+^{-1/2}\xi^0(\tau^3)^{-1/2}S^1\otimes\xi^+ \\ & \quad - q^{5/2}\lambda_+^{-1}\xi^0\sigma^2\otimes\xi^3 + q^{5/2}\lambda_+^{-1}\xi^0\tau^1\otimes\xi^3 \\ & \quad + q^{3/2}\lambda^2\lambda_+^{-1}\xi^0T^-T^2\otimes\xi^3 - q^{1/2}\lambda_+^{-1}\xi^0\sigma^2\otimes\xi^0 \\ & \quad - q^{5/2}\lambda_+^{-1}\xi^0\tau^1\otimes\xi^0 - q^{3/2}\lambda^2\lambda_+^{-1}\xi^0T^-T^2\otimes\xi^0 \\ & \quad + q^4\lambda^2\lambda_+^{-1/2}\xi^0(\tau^3)^{1/2}T^2\otimes\xi^- - q^{1/2}\xi^-(\tau^3)^{-1/2}\sigma^2\otimes\xi^+ \\ & \quad \left. + q^2\lambda\lambda_+^{-1/2}\xi^-T^2\otimes\xi^3 - q^2\lambda\lambda_+^{-1/2}\xi^-T^2\otimes\xi^0\right], \quad (31) \end{aligned}$$

and, similarly,

$$\begin{aligned} & -q^2c^{-1}\lambda\lambda_+^{-1}\eta_{\mu\nu}\xi^\mu\bar{\mathcal{L}}^\nu{}_\kappa\otimes\xi^\kappa = \\ & = c^{-1}q^{5/2}\lambda\lambda_+^{-1}\Lambda^{1/2}\left[\lambda\lambda_+^{-1/2}\xi^+(\tau^3)^{-1/2}S^1\otimes\xi^3 \right. \\ & \quad - \lambda\lambda_+^{-1/2}\xi^+(\tau^3)^{-1/2}S^1\otimes\xi^0 - q^{-1/2}\xi^+\tau^1\otimes\xi^- \\ & \quad - q^{-2}\lambda\lambda_+^{-1/2}\xi^3S^1\otimes\xi^+ + q^{-5/2}\lambda_+^{-1}\xi^3(\tau^3)^{1/2}\tau^1\otimes\xi^3 \\ & \quad + q^{-1/2}\lambda_+^{-1}\xi^3(\tau^3)^{-1/2}\sigma^2\otimes\xi^3 + q^{-3/2}\lambda^2\lambda_+^{-1}\xi^3(\tau^3)^{-1/2}T^+S^1\otimes\xi^3 \\ & \quad - q^{-1/2}\lambda_+^{-1}\xi^3(\tau^3)^{-1/2}\sigma^2\otimes\xi^0 + q^{-1/2}\lambda_+^{-1}\xi^3(\tau^3)^{1/2}\tau^1\otimes\xi^0 \\ & \quad \left. - q^{-3/2}\lambda^2\lambda_+^{-1}\xi^3(\tau^3)^{-1/2}T^+S^1\otimes\xi^0 - \lambda\lambda_+^{-1/2}\xi^3T^+\tau^1\otimes\xi^- \right] \end{aligned}$$

$$\begin{aligned}
& -q\lambda\lambda_+^{-1/2}\xi^3T^2\otimes\xi^- + q^{-2}\lambda\lambda_+^{-1/2}\xi^0S^1\otimes\xi^+ \\
& -q^{-5/2}\lambda_+^{-1}\xi^0(\tau^3)^{1/2}\tau^1\otimes\xi^3 + q^{-5/2}\lambda_+^{-1}\xi^0(\tau^3)^{-1/2}\sigma^2\otimes\xi^3 \\
& -q^{-3/2}\lambda^2\lambda_+^{-1}\xi^0(\tau^3)^{-1/2}T^+S^1\otimes\xi^3 - q^{-5/2}\lambda_+^{-1}\xi^0(\tau^3)^{-1/2}\sigma^2\otimes\xi^0 \\
& -q^{-1/2}\lambda_+^{-1}\xi^0(\tau^3)^{1/2}\tau^1\otimes\xi^0 + q^{-3/2}\lambda^2\lambda_+^{-1}\xi^0(\tau^3)^{-1/2}T^+S^1\otimes\xi^0 \\
& +\lambda\lambda_+^{-1/2}\xi^0T^+\tau^1\otimes\xi^- - q^{-1}\lambda\lambda_+^{-1/2}\xi^0T^2\otimes\xi^- \\
& -q^{-5/2}\xi^-\sigma^2\otimes\xi^+ + q^{-2}\lambda\lambda_+^{-1/2}\xi^-(\tau^3)^{-1/2}T^+\sigma^2\otimes\xi^3 \\
& +q^{-1}\lambda\lambda_+^{-1/2}\xi^-(\tau^3)^{1/2}T^2\otimes\xi^3 - q^{-2}\lambda\lambda_+^{-1/2}\xi^-(\tau^3)^{-1/2}T^+\sigma^2\otimes\xi^0 \\
& +q\lambda\lambda_+^{-1/2}\xi^-(\tau^3)^{1/2}T^2\otimes\xi^0 - q^{3/2}\lambda^2\xi^-T^+T^2\otimes\xi^-]. \tag{32}
\end{aligned}$$

Notice that the scaling operator  $\Lambda$  now has to satisfy the relations

$$\Lambda\mathbb{1} = q^4\mathbb{1}\Lambda, \quad \Lambda^{1/2}\xi^\mu = -q\xi^\mu\Lambda^{1/2}, \quad \mu \in \{+, 3, 0, -\}. \tag{33}$$

For later purpose, we wish to formulate a Clifford algebra for generators with lower indices. This Clifford algebra shall be determined by

$$(P_S)^{\mu\nu}{}_{\rho l}\tilde{\xi}_\mu\tilde{\xi}_\nu = 0, \quad (P_0)^{\mu\nu}{}_{\rho l}\tilde{\xi}_\mu\tilde{\xi}_\nu = \tilde{c}\eta_{\mu\nu}\mathbb{1}. \tag{34}$$

The above conditions are equivalent to

$$\tilde{\xi}_\mu\tilde{\xi}_\nu = -\hat{R}^{\rho l}{}_{\mu\nu}\tilde{\xi}_\rho\tilde{\xi}_l + \tilde{c}q\lambda_+\eta_{\mu\nu}\mathbb{1}, \tag{35}$$

$$\tilde{\xi}_\mu\tilde{\xi}_\nu = -(\hat{R}^{-1})^{\rho l}{}_{\mu\nu}\tilde{\xi}_\rho\tilde{\xi}_l + \tilde{c}q^{-1}\lambda_+\eta_{\mu\nu}\mathbb{1}. \tag{36}$$

More explicitly, we have

$$\begin{aligned}
& \tilde{\xi}_A\tilde{\xi}_A = 0, \quad A \in \{+, -\}, \\
& \tilde{\xi}_3\tilde{\xi}_\pm = -q^{\mp 2}\tilde{\xi}_\pm\tilde{\xi}_3, \\
& \tilde{\xi}_3\tilde{\xi}_3 = \lambda\tilde{\xi}_+\tilde{\xi}_- + q^2\tilde{c}\mathbb{1}, \\
& \tilde{\xi}_0\tilde{\xi}_\pm = -\tilde{\xi}_\pm\tilde{\xi}_0 \mp q^{\mp 1}\lambda\tilde{\xi}_\pm\tilde{\xi}_3, \\
& \tilde{\xi}_0\tilde{\xi}_3 = -\tilde{\xi}_3\tilde{\xi}_0 - \lambda\tilde{\xi}_+\tilde{\xi}_- - q\tilde{c}\lambda\mathbb{1}, \\
& \tilde{\xi}_0\tilde{\xi}_0 = -\tilde{c}\mathbb{1}, \\
& \tilde{\xi}_-\tilde{\xi}_+ = -\tilde{\xi}_+\tilde{\xi}_- - \tilde{c}\lambda_+\mathbb{1}. \tag{37}
\end{aligned}$$

In Sec. 3.2 we are going to deal with so-called '*inverse*'  $\gamma$ -matrices. Taking for  $\tilde{c}$  the same value as for  $c$  in (26), the '*inverse*'  $\gamma$ -matrices will then provide

a representation of the Clifford algebra (37), as can be checked by inserting the expressions for the 'inverse'  $\gamma$ -matrices into the relations in (37).

Finally, it should be mentioned that the relations in (37) are obtained by applying the replacements  $\xi^\mu \rightarrow \tilde{\xi}_\mu$  to the formulae in (26). However, the reader should have in mind that (37) describes an algebra being different from (26). Especially, we do *not* have  $\tilde{\xi}_\mu = \xi_\mu (\equiv \eta_{\mu\nu} \xi^\nu)$ . It should also be obvious that the identities in (27) remain valid under the substitutions  $\xi^\mu \rightarrow \tilde{\xi}_\mu$ .

### 3 $q$ -Deformed Dirac and Spin matrices

In this section we introduce  $q$ -analogs of  $\gamma$ -matrices and four-dimensional spin matrices ( $\Sigma$ -matrices). Moreover, we discuss some of their features and present useful formulae including these matrices. Our reasonings refer to four-dimensional  $q$ -deformed Euclidean space and  $q$ -deformed Minkowski space.

#### 3.1 $q$ -Deformed Euclidean space in four dimensions

##### 3.1.1 The $q$ -deformed Dirac matrices

In part I of the article we treated Pauli matrices for  $q$ -deformed Euclidean space in four dimensions. These Pauli matrices can be combined to give the  $\gamma$ -matrices of four-dimensional  $q$ -deformed Euclidean space (see also Refs. [42, 54–57]):

$$(\gamma^\mu)_a{}^b \equiv \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\tilde{\sigma}^\mu)_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad \mu = 1, \dots, 4. \quad (38)$$

In addition to this, we define

$$(\gamma^\mu)_{ab} \equiv \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\tilde{\sigma}^\mu)_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\gamma^\mu)^{ab} \equiv \begin{pmatrix} 0 & (\sigma^\mu)^{\alpha\dot{\beta}} \\ (\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix}. \quad (39)$$

Recalling the index properties of the Pauli matrices  $\sigma^\mu$  and  $\tilde{\sigma}^\mu$  [cf. part I], the above convention enables us to formulate the following rules for raising and lowering bispinor indices of  $\gamma$ -matrices:

$$(\gamma^\mu)^{ab} = (D_L^T)^{aa'} (\gamma^\mu)_{a'}{}^b, \quad (\gamma^\mu)_{ab} = (\gamma^\mu)_a{}^{b'} (D_R)_{b'b}, \quad (40)$$

where

$$(D_L)^{aa'} = \begin{pmatrix} \varepsilon^{\alpha\alpha'} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\alpha}'} \end{pmatrix}, \quad (D_R)_{bb'} = \begin{pmatrix} \varepsilon_{\beta'\beta} & 0 \\ 0 & \varepsilon_{\dot{\beta}'\dot{\beta}} \end{pmatrix}. \quad (41)$$

Notice that  $\varepsilon$  denotes the  $q$ -deformed spinor metric [see also App. C].

Inserting the expressions for the Pauli matrices  $\sigma^\mu$  and  $\tilde{\sigma}^\mu$  the  $\gamma$ -matrices in (38) become

$$\begin{aligned} (\gamma^1)_\alpha{}^\beta &= -q^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & (\gamma^2)_\alpha{}^\beta &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1/2} \\ q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ (\gamma^3)_\alpha{}^\beta &= \begin{pmatrix} 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -q^{-1/2} & 0 & 0 \end{pmatrix}, & (\gamma^4)_\alpha{}^\beta &= -q^{1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (42)$$

As can be verified by inserting, these  $\gamma$ -matrices fulfill the relations in (17), i.e. they establish a representation of the Clifford algebra to four-dimensional  $q$ -deformed Euclidean space. Interestingly, the matrices

$$(\gamma^\mu)_a{}^b = \begin{pmatrix} 0 & d(\sigma^\mu)_\alpha{}^{\dot{\beta}} \\ d^{-1}(\tilde{\sigma}^\mu)_\alpha{}^\beta & 0 \end{pmatrix}, \quad d \in \mathbb{C}, \quad (43)$$

give equally good representations of the Clifford generators. The reason for this lies in the fact that  $d$  cancels out against  $d^{-1}$  if we multiply two  $\gamma$ -matrices.

The Pauli matrices which went into the calculation of the above  $\gamma$ -matrices refer to symmetrized spinors. As we know from part I there are also Pauli matrices for antisymmetrized spinors. However, a little thought can show that the following results do not depend on the choice one makes in (38) for the Pauli matrices.

We can also find a  $q$ -analog of the matrix  $\gamma^5$ . With the help of the totally antisymmetric tensor of four-dimensional  $q$ -deformed Euclidean space (see App. C) we find:

$$\gamma^5 \equiv \frac{q}{[[2]]_{q^2} [[3]]_{q^2}} \varepsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (44)$$

Notice that the numerical factor was chosen in such a way that the result agrees with its undeformed counterpart. One readily checks that  $\gamma^5$  anticommutes with all  $\gamma$ -matrices:

$$\{\gamma^5, \gamma^\mu\} = 0, \quad \mu = 1, \dots, 4. \quad (45)$$

A short look at (44) tells us that  $\gamma^5$  has unit square, i.e.  $(\gamma^5)^2 = 1$ .

In analogy to the undeformed case there are several possibilities to build a basis of the Clifford algebra out of the  $\gamma$ -matrices (from now on the  $\gamma$ -matrices play the role of the Clifford generators). One basis is made up of antisymmetrized products of  $\gamma$ -matrices:

$$\begin{aligned} & \mathbb{1}, \gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^{[1}\gamma^2]_q, \gamma^{[1}\gamma^3]_q, \gamma^{[1}\gamma^4]_q, \gamma^{[2}\gamma^3]_q, \gamma^{[2}\gamma^4]_q, \\ & \gamma^{[3}\gamma^4]_q, \gamma^{[1}\gamma^2\gamma^3]_q, \gamma^{[1}\gamma^2\gamma^4]_q, \gamma^{[1}\gamma^3\gamma^4]_q, \gamma^{[2}\gamma^3\gamma^4]_q, \gamma^{[1}\gamma^2\gamma^3\gamma^4]_q. \end{aligned} \quad (46)$$

In what follows we will denote these elements in the order of their appearance by  $\Gamma^e \equiv \mathbb{1}, \Gamma^1, \Gamma^2, \dots, \Gamma^{1234}$ . Notice that the square brackets in (46) indicate antisymmetrization of vector indices in the sense of  $q$ -deformation. In this manner, we have

$$\begin{aligned} \gamma^{[\mu}\gamma^{\nu]}_q &\equiv (P_A)^{\mu\nu}{}_{\mu'\nu'} \gamma^{\mu'} \gamma^{\nu'}, \\ \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}_q &\equiv -\frac{1}{2q^2[[3]]_{q^2}} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu'\nu'\rho'\sigma} \gamma^{\rho'} \gamma^{\nu'} \gamma^{\mu'}, \\ \gamma^{[1}\gamma^2\gamma^3\gamma^4]_q &\equiv 2[[2]]_{q^2}^2 [[3]]_{q^2} \varepsilon^{1234} \varepsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu \\ &= 2q^{-1} [[2]]_{q^2}^3 [[3]]_{q^2}^2 \gamma^5, \end{aligned} \quad (47)$$

where  $P_A$  stands for the antisymmetrizer of the four-dimensional  $q$ -deformed Euclidean space.

Let us recall that in part I of the article we introduced a quantum trace for Pauli matrices. This notion can be extended to products of  $\gamma$ -matrices as follows:

$$\text{Tr}_q \gamma^\mu \gamma^\nu \equiv \text{Tr} D_L^T \gamma^\mu \gamma^\nu D_R. \quad (48)$$

where usual matrix multiplication is understood on the right-hand side of (48).

The quantum trace can be viewed as a sesquilinear form. Thus, it seems reasonable to seek matrices  $\Gamma_{l,A}$ ,  $A \in \{e, 1, 2, \dots, 1234\}$ , with the property

$$\text{Tr}_q (\Gamma_{l,A} \Gamma^B) = \delta^B_A. \quad (49)$$

This requirement defines a left-dual basis of the Clifford algebra with basis elements  $\Gamma^B$ . For the elements  $\Gamma_{l,A}$  of the left-dual basis we found the expressions

$$\begin{aligned}\Gamma_{l,e} &= -\frac{1}{2}\lambda_+^{-1}\Gamma^e, & \Gamma_{l,1} &= -\frac{1}{2}q^{-1}\Gamma^4, & \Gamma_{l,2} &= -\frac{1}{2}\Gamma^3, \\ \Gamma_{l,3} &= -\frac{1}{2}\Gamma^2, & \Gamma_{l,4} &= -\frac{1}{2}q\Gamma^1,\end{aligned}\tag{50}$$

$$\begin{aligned}\Gamma_{l,12} &= -\Gamma^{34}, & \Gamma_{l,13} &= -\Gamma^{24}, \\ \Gamma_{l,14} &= -q^{-2}(q^4 + 1)\lambda_+^{-1}\Gamma^{14} + \lambda\lambda_+^{-1}\Gamma^{23}, \\ \Gamma_{l,23} &= -2\lambda_+^{-1}\Gamma^{23} + \lambda\lambda_+^{-1}\Gamma^{14}, \\ \Gamma_{l,24} &= -q^2\Gamma^{13}, & \Gamma_{34} &= -q^2\Gamma^{12},\end{aligned}\tag{51}$$

$$\begin{aligned}\Gamma_{l,123} &= -2q\Gamma^{234}, & \Gamma_{l,124} &= -2q^2\Gamma^{134}, & \Gamma_{l,134} &= -2q^2\Gamma^{124}, \\ \Gamma_{l,234} &= -2q^3\Gamma^{123}, & \Gamma_{l,1234} &= -\frac{1}{8}q^3[[2]]_{q^2}^{-7}[[3]]_{q^2}^{-4}\Gamma^{1234}.\end{aligned}\tag{52}$$

Solving these equations for the  $\Gamma^B$ ,  $B \in \{e, 1, 2, \dots, 1234\}$ , shows that the  $\Gamma_{l,A}$  indeed establish a basis of the Clifford algebra to four-dimensional  $q$ -deformed Euclidean space.

Alternatively, we can ask for matrices  $\Gamma_{r,A}$ ,  $A \in \{e, 1, 2, \dots, 1234\}$ , subject to

$$\text{Tr}_q(\Gamma^B \Gamma_{r,A}) = \delta^B_A.\tag{53}$$

These matrices lead to a right-dual basis. We found that the  $\Gamma_{r,A}$  are identical to the  $\Gamma_{l,A}$  with the exception of the following matrices:

$$\begin{aligned}\Gamma_{r,1} &= q^2\Gamma_{l,1}, & \Gamma_{r,4} &= q^{-2}\Gamma_{l,4}, \\ \Gamma_{r,12} &= q^2\Gamma_{l,12}, & \Gamma_{r,13} &= q^2\Gamma_{l,13}, & \Gamma_{r,24} &= q^{-2}\Gamma_{l,24}, & \Gamma_{r,34} &= q^{-2}\Gamma_{l,34}, \\ \Gamma_{r,123} &= q^2\Gamma_{l,123}, & \Gamma_{r,234} &= q^{-2}\Gamma_{l,234}.\end{aligned}\tag{54}$$

The reader may wonder why the two dual bases are not the same. This has to do with the fact that the quantum metric is not symmetrical.

As we know, the 16 matrices  $\Gamma^A$ ,  $A \in \{e, 1, 2, \dots, 1234\}$ , constitute a basis. Thus, we should be able to expand an arbitrary  $4 \times 4$  matrix  $M$  in terms of these matrices, i.e.

$$M = \sum_A c_A \Gamma^A.\tag{55}$$



In analogy to the undeformed case, each expansion coefficient is given by a quantum trace over the product of  $M$  with a dual matrix:

$$c_B = \text{Tr}_q(M \Gamma_{r,B}) = \text{Tr}_q(\Gamma_{l,B} M). \quad (56)$$

Plugging these expressions into the expansion of  $M$  we can finally read off as completeness relations

$$\begin{aligned} \delta_a^b \delta_c^d &= \sum_B (D_R)_{cm} (D_L^T)^{mk} (\Gamma_{l,B})_k^b (\Gamma^B)_a^d \\ &= \sum_B (\Gamma^B)_a^d (\Gamma_{r,B})_c^k (D_R)_{km} (D_L^T)^{mb} \end{aligned} \quad (57)$$

with the matrices  $D_R$  and  $D_L$  from (41). To write this result in a more convenient form we introduce

$$\begin{aligned} (\Gamma_{l,B}^c)_a^b &\equiv (D_R)_{am} (D_L^T)^{mk} (\Gamma_{l,B})_k^b, \\ (\Gamma_{l,B}^c)_a^b &\equiv (\Gamma_{r,B})_a^k (D_R)_{km} (D_L^T)^{mb}. \end{aligned} \quad (58)$$

With these new matrices at hand we finally arrive at

$$\sum_B (\Gamma_{l,B}^c)_c^b (\Gamma^B)_a^d = \delta_a^b \delta_c^d, \quad \sum_B (\Gamma^B)_a^d (\Gamma_{r,B}^c)_c^b = \delta_a^b \delta_c^d. \quad (59)$$

For the sake of completeness we would like to express the matrices  $\Gamma_{l,B}^c$  in terms of the  $\Gamma^B$ :

$$\begin{aligned} \Gamma_e^c &= \frac{1}{2}(q^4 + 1)q^{-2}\lambda_+^{-2} \Gamma^e - \frac{1}{2}\lambda\lambda_+^{-1} \Gamma^{14}, \\ \Gamma_1^c &= \frac{1}{2} \Gamma^4, \quad \Gamma_2^c = \frac{1}{4}\lambda_+ \Gamma^3 + \frac{q}{2}\lambda \Gamma^{134}, \\ \Gamma_3^c &= \frac{1}{4}\lambda_+ \Gamma^2 + \frac{q}{2}\lambda \Gamma^{124}, \quad \Gamma_4^c = \frac{1}{2} \Gamma^1, \end{aligned} \quad (60)$$

$$\begin{aligned} \Gamma_{12}^c &= q \Gamma^{34}, \quad \Gamma_{13}^c = q \Gamma^{24}, \\ \Gamma_{14}^c &= 2q^{-2}(q^4 + 1)\lambda_+^{-2} \Gamma^{14} - 2\lambda\lambda_+^{-2} \Gamma^{23} - 2\lambda\lambda_+^{-1} \Gamma^e, \\ &\quad + \frac{1}{4}q^3\lambda^2[[2]]_{q^2}^{-5}[[2]]_{q^2}^{-2} \Gamma^{1234}, \\ \Gamma_{23}^c &= 4\lambda_+^{-2} \Gamma^{23} - 2\lambda\lambda_+^{-2} \Gamma^{14} - \frac{1}{2}q^3\lambda[[2]]_{q^2}^{-5}[[3]]_{q^2}^{-2} \Gamma^{1234}, \end{aligned}$$

$$\Gamma_{24}^c = q \Gamma^{13}, \quad \Gamma_{34}^c = q \Gamma^{12}, \quad (61)$$

$$\begin{aligned} \Gamma_{123}^c &= 2q^2 \Gamma^{234}, \quad \Gamma_{124}^c = q^2 \lambda_+ \Gamma^{134} + \frac{1}{2} q \lambda \Gamma^3, \\ \Gamma_{134}^c &= q^2 \lambda_+ \Gamma^{124} + \frac{1}{2} q \lambda \Gamma^2, \quad \Gamma_{234}^c = 2q^2 \Gamma^{123}, \end{aligned} \quad (62)$$

$$\begin{aligned} \Gamma_{1234}^c &= \frac{1}{8} q^2 (q^4 + 1) [[2]]_{q^2}^{-8} [[3]]_{q^2}^{-4} \Gamma^{1234} \\ &\quad + \frac{1}{2} q^3 \lambda [[2]]_{q^2}^{-5} [[3]]_{q^2}^{-2} \left( \frac{1}{2} \lambda \Gamma^{14} - \Gamma^{23} \right). \end{aligned} \quad (63)$$

Similar formulae hold for the matrices  $\Gamma_{r,B}^c$

Now, we turn to a second basis of the Clifford algebra under consideration. It consists of the elements

$$\begin{aligned} &\mathbb{1}, \gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^5 \gamma^1, \gamma^5 \gamma^2, \gamma^5 \gamma^3, \gamma^5 \gamma^4, \\ &\gamma^{[1} \gamma^{2]}_q, \gamma^{[1} \gamma^{3]}_q, \gamma^{[1} \gamma^{4]}_q, \gamma^{[2} \gamma^{3]}_q, \gamma^{[2} \gamma^{4]}_q, \gamma^{[3} \gamma^{4]}_q. \end{aligned} \quad (64)$$

Instead of antisymmetrized products of three or four  $\gamma$ -matrices the new basis contains the matrices  $\Gamma^{5\mu} \equiv \gamma^5 \gamma^\mu$ ,  $\mu = 1, \dots, 4$ , and  $\Gamma^5 \equiv \gamma^5$ . That (64) is indeed equivalent to our previous basis should become obvious from the relations

$$\begin{aligned} \gamma^{[\mu} \gamma^\nu \gamma^{\rho]}_q &= \frac{1}{2} q^{-1} [[2]]_{q^2}^{-1} [[3]]_{q^2}^{-1} \varepsilon^{\mu\nu\rho\sigma} g_{\sigma\lambda} \gamma^\lambda \gamma^5, \\ \gamma^5 \gamma^\rho &= \lambda_+ \varepsilon^{\rho\lambda\nu\mu} g_{\mu\mu'} g_{\nu\nu'} g_{\lambda\lambda'} \gamma^{[\mu'} \gamma^{\nu'} \gamma^{\lambda']}_q \\ &= -[[3]]_{q^2}^{-1} \varepsilon_{\sigma\lambda\rho\nu} g^{\sigma\mu} \gamma^\nu \gamma^\rho \gamma^\lambda. \end{aligned} \quad (65)$$

The above reasonings about left- and right-duals carry over to the new basis. There are only some minor changes concerning the matrices  $\Gamma^5$  and  $\Gamma^{5\mu}$ . In this respect, we have to notice that their left-duals read as

$$\begin{aligned} \Gamma_{l,5} &= -\frac{1}{2} \lambda_+^{-1} \Gamma^5 = -\frac{1}{2} \lambda_+^{-1} \gamma^5, \\ \Gamma_{l,51} &= \frac{1}{2} q^{-1} \Gamma^{54}, \quad \Gamma_{l,52} = \frac{1}{2} \Gamma^{52}, \\ \Gamma_{l,53} &= \frac{1}{2} \Gamma^{52}, \quad \Gamma_{l,54} = \frac{1}{2} q \Gamma^{51}, \end{aligned} \quad (66)$$

and the corresponding right-duals are of the same form apart from

$$\Gamma_{r,51} = q^2 \Gamma_{51}, \quad \Gamma_{r,54} = q^{-2} \Gamma_{54}. \quad (67)$$

It should be emphasized that the formulae (55)-(59) also apply to the basis in (64) if we take into account that

$$\Gamma_2^c = \frac{1}{4} \lambda_+ \Gamma^3 + \frac{1}{4} \lambda \Gamma^{53}, \quad \Gamma_3^c = \frac{1}{4} \lambda_+ - \frac{1}{4} \lambda \Gamma^{52}, \quad (68)$$

$$\begin{aligned} \Gamma_{14}^c &= 2q^{-2}(q^4 + 1)\lambda_+^{-2} \Gamma^{14} - 2\lambda\lambda_+^{-2} \Gamma^{23} + \frac{1}{2}\lambda^2\lambda_+^{-2} \Gamma^5 - \frac{1}{2}\lambda\lambda_+^{-1} \Gamma^e, \\ \Gamma_{23}^c &= 4\lambda_+^{-2} \Gamma^{23} - 2\lambda\lambda_+^{-1} \Gamma^{14} - \lambda^2\lambda_+^{-2} \Gamma^5, \end{aligned} \quad (69)$$

$$\begin{aligned} \Gamma_{51}^c &= -\frac{1}{2} \Gamma^{54}, \quad \Gamma_{52}^c = -\frac{1}{4} \lambda_+ \Gamma^{53} - \frac{1}{4} \lambda \Gamma^3, \\ \Gamma_{53}^c &= -\frac{1}{4} \lambda_+ \Gamma^{52} - \frac{1}{4} \lambda \Gamma^2, \quad \Gamma_{54}^c = -\frac{1}{2} \Gamma^{51}, \end{aligned} \quad (70)$$

$$\Gamma_5^c = \frac{1}{2} q^{-2}(q^4 + 1) \Gamma^5 + \lambda^2 \lambda_+^{-2} \left( \frac{1}{2} \Gamma^{14} - \Gamma^{23} \right), \quad (71)$$

whereas the expressions for the remaining matrices  $\Gamma_A^c$  again follow from (60) and (61).

### 3.1.2 Relations concerning $q$ -deformed Dirac matrices

Now, we would like to present  $q$ -analogs of well-known relations including  $\gamma$ -matrices. Let us first note that in the remainder of this section multiplication of  $\gamma$ -matrices is always understood as

$$\gamma^\mu \gamma^\nu = (\gamma^\mu \gamma^\nu)_a{}^b = (\gamma^\mu)_a{}^{a'} (\gamma^\nu)_{a'}{}^b. \quad (72)$$

Computations in quantum field theory often require to evaluate traces over products of  $\gamma$ -matrices. We found as  $q$ -analogs of well-known trace relations:

$$\begin{aligned} \text{Tr}_q \gamma^\mu &= 0, \quad \text{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho = 0, \quad \text{Tr}_q \gamma^\mu \gamma^\nu = 2\eta^{\mu\nu}, \\ \text{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma &= -q g^{\mu\nu} g^{\rho\sigma} + g^{\mu\kappa} \hat{R}^{\nu\rho}{}_{\kappa\lambda} g^{\lambda\sigma} - q^{-1} g^{\mu\sigma} g^{\nu\rho} \\ &= -q^{-1} g^{\mu\nu} g^{\rho\sigma} + g^{\mu\kappa} (\hat{R}^{-1})^{\nu\rho}{}_{\kappa\lambda} g^{\lambda\sigma} - q g^{\mu\sigma} g^{\nu\rho}, \end{aligned} \quad (73)$$

for all  $\mu, \nu, \rho, \sigma \in \{1, 2, 3, 4\}$ . Furthermore, we have

$$\begin{aligned}\mathrm{Tr}_q \gamma^5 &= \mathrm{Tr}_q \gamma^\mu \gamma^5 = \mathrm{Tr}_q \gamma^\mu \gamma^\nu \gamma^5 = \mathrm{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^5 = 0, \\ \mathrm{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 &= -q^{-2} \varepsilon^{\mu\nu\rho\sigma},\end{aligned}\tag{74}$$

and

$$\begin{aligned}\mathrm{Tr}_q \gamma^5 \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} \gamma^{\mu_6} &= \\ &= -2q^{-1} [k_1 \eta^{\mu_1 \mu_2} \varepsilon^{\mu_3 \mu_4 \mu_5 \mu_6} + k_2 \eta^{\mu_1 \mu'_3} \hat{R}^{\mu_2 \mu_3}{}_{\mu'_3 \mu'_2} \varepsilon^{\mu'_2 \mu_4 \mu_5 \mu_6} \\ &\quad + k_3 \eta^{\mu_2 \mu_3} \varepsilon^{\mu_1 \mu_4 \mu_5 \mu_6} + k_4 \eta^{\mu_4 \mu_5} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_6} \\ &\quad + k_5 \eta^{\mu_4 \mu'_6} \hat{R}^{\mu_5 \mu_6}{}_{\mu'_6 \mu'_5} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu'_5} + k_6 \eta^{\mu_5 \mu_6} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4}],\end{aligned}\tag{75}$$

where

$$k_1 = 1, \quad k_2 = -q^{-1}, \quad k_3 = q^{-2}, \quad k_4 = 1, \quad k_5 = -q^{-1}, \quad k_6 = q^{-2}.\tag{76}$$

If we replace  $\hat{R}$  with  $\hat{R}^{-1}$  the coefficients in (75) instead take on the values

$$k_1 = q^{-2}, \quad k_2 = -q^{-1}, \quad k_3 = 1, \quad k_4 = q^{-2}, \quad k_5 = -q^{-1}, \quad k_6 = 1.\tag{77}$$

It is also worth recording here that quantum traces over products of  $\gamma$ -matrices obey a kind of cyclicity:

$$\begin{aligned}\mathrm{Tr}_q \gamma^\mu \gamma^\nu &= q^3 \hat{R}^{\mu\nu}{}_{\nu' \mu'} \mathrm{Tr}_q \gamma^{\nu'} \gamma^{\mu'}, \\ \mathrm{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma &= q^3 \hat{R}^{\mu\nu}{}_{\nu' \mu'} \hat{R}^{\mu' \rho}{}_{\rho' \mu''} \hat{R}^{\mu'' \sigma}{}_{\sigma' \mu'''} \mathrm{Tr}_q \gamma^{\nu'} \gamma^{\rho'} \gamma^{\sigma'} \gamma^{\mu'''}. \end{aligned}\tag{78}$$

Applying the substitutions  $\hat{R} \rightarrow \hat{R}^{-1}$ ,  $q \rightarrow q^{-1}$  to the expressions in (78) leads us to alternative formulations.

Next, we turn to formulae with  $\gamma$ -matrices being contracted by the quantum metric:

$$\begin{aligned}g_{\mu\nu} \gamma^\mu \gamma^\nu &= \lambda_+ \mathbb{1}, \\ g_{\rho\mu} \hat{R}^{\mu\nu}{}_{\nu' \mu'} \gamma^\rho \gamma^{\nu'} \gamma^{\mu'} &= -\gamma^\nu, \\ g_{\sigma\mu} \hat{R}^{\mu\nu}{}_{\nu' \mu'} \hat{R}^{\mu' \rho}{}_{\rho' \mu''} \gamma^\sigma \gamma^{\nu'} \gamma^{\rho'} \gamma^{\mu''} &= g^{\nu\rho} \mathbb{1}.\end{aligned}\tag{79}$$

These equations remain valid if we perform the substitution  $\hat{R} \rightarrow \hat{R}^{-1}$ . Additionally, we have the little bit lengthy relation

$$g_{\delta\mu} \hat{R}^{\mu\nu}{}_{\nu' \mu'} \hat{R}^{\mu' \rho}{}_{\rho' \mu''} \hat{R}^{\mu'' \sigma}{}_{\sigma' \mu'''} \gamma^\delta \gamma^{\nu'} \gamma^{\rho'} \gamma^{\sigma'} \gamma^{\mu'''}$$

$$= -q^3 \hat{R}^{\rho''\sigma''}{}_{\sigma'\rho'} \hat{R}^{\nu''\sigma}{}_{\sigma''\nu'} \hat{R}^{\nu\rho}{}_{\rho''\nu''} \gamma^{\sigma'} \gamma^{\rho'} \gamma^{\nu'}, \quad (80)$$

which can be recognized as  $q$ -analog of  $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$ . Once again, the replacements  $\hat{R} \rightarrow \hat{R}^{-1}$ ,  $q^{-3} \rightarrow -q^3$  lead us to an equally good relation.

We close with a decomposition formula for products of three  $\gamma$ -matrices. Concretely, we have

$$\begin{aligned} \gamma^\mu \gamma^\nu \gamma^\rho &= \gamma^{[\mu} \gamma^\nu \gamma^{\rho]}_q + \frac{q}{2} \eta^{\mu\nu} \gamma^\rho \\ &\quad - \frac{1}{2} \eta^{\mu\lambda'} \hat{R}^{\nu\rho}{}_{\lambda'\nu'} \gamma^{\nu'} + \frac{1}{2q} \gamma^\mu \eta^{\nu\rho}. \end{aligned} \quad (81)$$

There is also a version of this formula with  $\hat{R}$  and  $q$  being replaced by  $\hat{R}^{-1}$  and  $q^{-1}$ , respectively. The above result tells us that products of three  $\gamma$ -matrices split into a totally antisymmetric part and a linear combination of  $\gamma$ -matrices.

### 3.1.3 The $q$ -deformed spin matrices and some more relations

It is now our aim to discuss  $q$ -analogs of the four-dimensional  $\Sigma$ -matrices. In analogy to the undeformed case they are defined by

$$(\Sigma^{\mu\nu})_a{}^b = (P_A)^{\mu\nu}{}_{\mu'\nu'} (\gamma^\mu)_a{}^{a'} (\gamma^\nu)_{a'}{}^b. \quad (82)$$

The four-dimensional  $\Sigma$ -matrices are made up of their two-dimensional counterparts  $\sigma^{\mu\nu}$  and  $\tilde{\sigma}^{\mu\nu}$  according to

$$(\Sigma^{\mu\nu})_a{}^b = \begin{pmatrix} (\sigma^{\mu\nu})_{\alpha\beta} & 0 \\ 0 & (\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \quad (83)$$

Now, we come to useful relations involving  $\gamma$ - and  $\Sigma$ -matrices. Making suitable ansatz we could prove the following identities:

$$\begin{aligned} \gamma^\mu \gamma^\nu &= \Sigma^{\mu\nu} + \lambda_+^{-1} g^{\mu\nu}, \\ \gamma^5 \gamma^\mu \gamma^\nu &= \lambda_+^{-1} g^{\mu\nu} \gamma^5 + q^{-1} \lambda_+^{-1} \varepsilon^{\mu\nu\rho\sigma} g_{\rho\rho'} g_{\sigma\sigma'} \Sigma^{\sigma'\rho'}, \\ \gamma^5 \Sigma^{\mu\nu} &= q^{-1} \lambda_+^{-1} \varepsilon^{\mu\nu\rho\sigma} g_{\rho\rho'} g_{\sigma\sigma'} \Sigma^{\sigma'\rho'}, \\ \gamma^\rho \Sigma^{\mu\nu} &= \frac{1}{2} q^{-2} \varepsilon^{\rho\mu\nu\lambda} g_{\lambda\delta} \gamma^5 \gamma^\delta + \frac{1}{2} \lambda_+ (P_A)^{\mu\nu}{}_{\mu'\nu'} g^{\rho\mu'} \gamma^{\nu'}, \end{aligned} \quad (84)$$

$$\Sigma^{\mu\nu}\gamma^\rho = \frac{1}{2}q^{-2}\varepsilon^{\mu\nu\rho\lambda}g_{\lambda\delta}\gamma^5\gamma^\delta + \frac{1}{2}\lambda_+(P_A)^{\mu\nu}{}_{\mu'\nu'}g^{\nu'\rho}\gamma^{\mu'}, \quad (85)$$

$$\begin{aligned} \gamma^5\gamma^\rho\Sigma^{\mu\nu} &= \frac{1}{2}q^{-2}\varepsilon^{\rho\mu\nu\lambda}g_{\lambda\delta}\gamma^\delta + \frac{1}{2}\lambda_+(P_A)^{\mu\nu}{}_{\mu'\nu'}g^{\rho\mu'}\gamma^5\gamma^{\nu'}, \\ \Sigma^{\mu\nu}\gamma^5\gamma^\rho &= \frac{1}{2}q^{-2}\varepsilon^{\mu\nu\rho\lambda}g_{\lambda\delta}\gamma^\delta + \frac{1}{2}\lambda_+(P_A)^{\mu\nu}{}_{\mu'\nu'}g^{\nu'\rho}\gamma^5\gamma^{\mu'}, \end{aligned} \quad (86)$$

$$\begin{aligned} \Sigma^{\mu\nu}\Sigma^{\kappa\lambda} &= \frac{1}{2}(P_A)^{\mu\nu}{}_{\mu'\nu'}(P_A)^{\kappa\lambda}{}_{\kappa'\lambda'}g^{\mu'\lambda'}g^{\nu'\kappa'} + \frac{1}{2}q^{-2}\lambda_+^{-1}\varepsilon^{\mu\nu\kappa\lambda}\gamma^5 \\ &\quad + \lambda_+(P_A)^{\mu\nu}{}_{\mu'\nu'}(P_A)^{\kappa\lambda}{}_{\kappa'\lambda'}g^{\nu'\kappa'}\Sigma^{\mu'\lambda'}. \end{aligned} \quad (87)$$

For quantum traces over  $\Sigma$ -matrices we have

$$\begin{aligned} \text{Tr}_q \Sigma^{\mu\nu} &= 0, \\ \text{Tr}_q \Sigma^{\rho\sigma}\Sigma^{\mu\nu} &= q^{-1}\lambda_+(P_A)^{\mu\nu}{}_{\mu'\nu'}\hat{R}^{\sigma\mu'}{}_{\mu''\sigma'}g^{\rho\mu''}g^{\sigma'\nu'}, \end{aligned} \quad (88)$$

where we are again allowed to apply the replacements  $\hat{R} \rightarrow \hat{R}^{-1}$ ,  $q \rightarrow q^{-1}$ .

Next we would like to present  $q$ -deformed analoga of some relations listed in Ref. [48] (see p.13):

$$g_{\mu\mu'}\hat{R}^{\mu'\nu}{}_{\nu'\mu''}\hat{R}^{\mu''\rho}{}_{\rho'\mu'''}\gamma^\mu\Sigma^{\nu'\rho'}\gamma^{\mu'''} = 0, \quad (89)$$

$$\begin{aligned} \gamma^\mu\Sigma^{\nu'\rho'}\gamma^{\sigma'}\gamma^{\mu''''}\hat{R}^{\mu'\nu}{}_{\nu'\mu''}\hat{R}^{\mu''\rho}{}_{\rho'\mu'''}\hat{R}^{\mu'''\sigma}{}_{\sigma'\mu''''}g_{\mu\mu'} &= \\ &= q^2\hat{R}^{\nu\sigma'}{}_{\sigma''\nu'}\hat{R}^{\rho\sigma}{}_{\sigma'\rho'}\gamma^{\sigma''}\Sigma^{\nu'\rho'}, \\ \gamma^\mu\gamma^{\nu'}\Sigma^{\rho'\sigma'}\gamma^{\mu''''}\hat{R}^{\mu'\nu}{}_{\nu'\mu''}\hat{R}^{\mu''\rho}{}_{\rho'\mu'''}\hat{R}^{\mu'''\sigma}{}_{\sigma'\mu''''}g_{\mu\mu'} &= \\ &= q^2\hat{R}^{\nu'\sigma}{}_{\sigma'\nu''}\hat{R}^{\nu\rho}{}_{\rho'\nu'}\Sigma^{\rho'\sigma'}\gamma^{\nu''}, \end{aligned} \quad (90)$$

$$\begin{aligned} \gamma^\mu\gamma^{\rho'_1}\gamma^{\rho'_2}\gamma^{\rho'_3}\gamma^{\rho'_4}\gamma^\nu\hat{R}^{\mu'\rho_1}{}_{\rho'_1\mu''}\hat{R}^{\mu''\rho_2}{}_{\rho'_2\mu'''}\hat{R}^{\mu'''\rho_3}{}_{\rho'_3\mu''''}\hat{R}^{\mu''''\rho_4}{}_{\rho'_4\nu}g_{\mu\mu'} &= \\ &= q^2\gamma^{\rho'_4'''}\gamma^{\rho'_1}\gamma^{\rho'_2}\gamma^{\rho'_3}\hat{R}^{\rho_1\rho'_4}{}_{\rho'_4'''\rho'_1}\hat{R}^{\rho_2\rho'_4}{}_{\rho'_4''\rho'_2}\hat{R}^{\rho_3\rho_4}{}_{\rho'_4\rho'_3} \\ &\quad + q^4\gamma^{\rho'_2}\gamma^{\rho'_3}\gamma^{\rho'_4}\gamma^{\rho'_1'''}\hat{R}^{\rho_1\rho_2}{}_{\rho'_2\rho'_1}\hat{R}^{\rho_1\rho_3}{}_{\rho'_3\rho'_1}\hat{R}^{\rho_1\rho_4}{}_{\rho'_4\rho'_1'''} \\ &= q^2\gamma^{\rho'_3''}\gamma^{\rho'_2''}\gamma^{\rho'_1''}\gamma^{\rho_4}\hat{R}^{\rho_2\rho'_3}{}_{\rho'_3\rho'_2''}\hat{R}^{\rho_1\rho_2}{}_{\rho'_2\rho'_1}\hat{R}^{\rho_1\rho_3}{}_{\rho'_3\rho'_1''} \\ &\quad + q^4\gamma^{\rho_1}\gamma^{\rho'_4''}\gamma^{\rho'_3''}\gamma^{\rho'_2''}\hat{R}^{\rho_2\rho'_4}{}_{\rho'_4''\rho'_2}\hat{R}^{\rho_2\rho'_3}{}_{\rho'_3\rho'_2''}\hat{R}^{\rho_3\rho_4}{}_{\rho'_4\rho'_3}, \end{aligned} \quad (91)$$

where the substitutions  $\hat{R} \rightarrow \hat{R}^{-1}$ ,  $q \rightarrow q^{-1}$  lead to further relations.

It should also be mentioned that a product of a  $\Sigma$ -matrix with  $\gamma^5$  can be written as linear combination of  $\Sigma$ -matrices. For the independent  $\Sigma$ -matrices we have

$$\begin{aligned}\Sigma^{12}\gamma^5 &= \Sigma^{12}, & \Sigma^{13}\gamma^5 &= -\Sigma^{13}, \\ \Sigma^{14}\gamma^5 &= 2\lambda_+^{-1}\Sigma^{23} - \lambda\lambda_+^{-1}\Sigma^{14}, \\ \Sigma^{23}\gamma^5 &= 2\lambda_+^{-1}\Sigma^{14} + \lambda\lambda_+^{-1}\Sigma^{23}, \\ \Sigma^{24}\gamma^5 &= -\Sigma^{24}, & \Sigma^{34}\gamma^5 &= \Sigma^{34}.\end{aligned}\tag{92}$$

As in the undeformed case the scalar 1, the vector  $\gamma^\mu$ , the pseudoscalar  $\gamma^5$ , the 'axial' vector  $\gamma^5\gamma^\mu$ , and the antisymmetric tensor  $\Sigma^{\mu\nu}$  can be taken as 16 independent  $4 \times 4$  matrices. In this manner, we have the following expansion for an arbitrary  $4 \times 4$  matrix  $A$ :

$$A = a_0\mathbb{1} + a_5\gamma^5 + v_\mu\gamma^\mu + a_\mu\gamma^5\gamma^\mu + T_{\mu\nu}\Sigma^{\mu\nu},\tag{93}$$

where

$$\begin{aligned}a_0 &= \frac{1}{2}\lambda_+^{-1}\text{Tr}_q(A), \\ a_5 &= \frac{1}{2}\lambda_+^{-1}\text{Tr}_q(\gamma^5 A), \\ v_\mu &= \frac{1}{2}g_{\mu\nu}\text{Tr}_q(\gamma^\nu A), \\ a_\mu &= -\frac{1}{2}g_{\mu\nu}\text{Tr}_q(\gamma^5\gamma^\nu A), \\ T_{\mu\nu} &= \lambda_+^{-1}g_{\mu\mu'}g_{\nu\nu'}\text{Tr}_q(\Sigma^{\nu'\mu'}A).\end{aligned}\tag{94}$$

If we use instead the matrices in (46) as basis the expansion reads

$$A = a_0\mathbb{1} + a_{16}\Gamma^{1234} + v_\mu\gamma^\mu + a_K\Gamma^K + T_{\mu\nu}\Sigma^{\mu\nu},\tag{95}$$

where

$$\begin{aligned}a_0 &= \frac{1}{2}\lambda_+^{-1}\text{Tr}_q(A), \\ a_{16} &= \frac{1}{8}q^3[[2]]_{q^2}^{-7}[[3]]_{q^2}^{-4}\text{Tr}_q(\Gamma^{1234}A), \\ v_\mu &= \frac{1}{2}g_{\mu\nu}\text{Tr}_q(\gamma^\nu A), \\ a_K &= -\text{Tr}_q(\Gamma_K A), \\ T_{\mu\nu} &= \lambda_+^{-1}g_{\mu\mu'}g_{\nu\nu'}\text{Tr}_q(\Sigma^{\nu'\mu'}A),\end{aligned}\tag{96}$$

and  $K$  is an index running over  $K \in \{123, 124, 134, 234\}$ .

## 3.2 $q$ -Deformed Minkowski space

In this section we deal with  $\gamma$ - and  $\Sigma$ -matrices for  $q$ -deformed Minkowski space. Our reasonings are rather similar to those for the Euclidean case. However, there is one remarkable difference compared to the presentation in the previous section. It stems from the fact that we now have to distinguish between  $\gamma$ -matrices and their so-called 'inverse' variants, which should not be confused with inverse matrices in the sense of matrix multiplication.

### 3.2.1 The $q$ -deformed Dirac matrices

**Definition of  $\gamma$ -matrices:** The  $\gamma$ -matrices are defined as

$$(\gamma^\mu)_a{}^b \equiv \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (97)$$

where  $\sigma^\mu$  and  $\bar{\sigma}^\mu$  denote the Pauli matrices of  $q$ -deformed Minkowski space, as they were introduced in part I of the article. The  $\gamma$ -matrices with two upper indices and those with two lower indices are given by

$$(\gamma^\mu)_{ab} \equiv \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\gamma^\mu)^{ab} \equiv \begin{pmatrix} 0 & (\sigma^\mu)^{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix}. \quad (98)$$

Again, we can find matrices for raising bispinor indices of  $\gamma$ -matrices. In view of the properties of Pauli matrices for  $q$ -deformed Minkowski space we should have

$$(\gamma^\mu)^{ab} = (D_L^T)^{aa'} (\gamma^\mu)_{a'}{}^b, \quad (\gamma^\mu)_{ab} = (\gamma^\mu)_a{}^{b'} (D_R)_{b'b}, \quad (99)$$

with

$$(D_L)^{aa'} = \begin{pmatrix} \varepsilon^{\alpha\alpha'} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\alpha}'} \end{pmatrix}, \quad (D_R)_{bb'} = \begin{pmatrix} \varepsilon_{\beta\beta'} & 0 \\ 0 & -\varepsilon_{\dot{\beta}\dot{\beta}'} \end{pmatrix}. \quad (100)$$

Choosing Pauli matrices for symmetrized spinors [cf. part I] the  $\gamma$ -matrices in (97) explicitly read as

$$(\gamma^+)_{\alpha}{}^{\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{3/2} & 0 \\ 0 & 0 & 0 & 0 \\ q^{-1/2} & 0 & 0 & 0 \end{pmatrix}, \quad (\gamma^-)_{\alpha}{}^{\beta} = \begin{pmatrix} 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 \\ 0 & -q^{-3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$



$$(\gamma^3)_\alpha{}^\beta = \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & -q^2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -q^{-2} & 0 & 0 \end{pmatrix}, (\gamma^0)_\alpha{}^\beta = \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (101)$$

These  $\gamma$ -matrices fulfill the Clifford algebra relations in (26) as can be checked by inserting.

The matrices in (101) enable us to calculate a  $q$ -analog of the matrix  $\gamma^5$ . It is defined as the totally antisymmetric product of four  $\gamma$ -matrices. In this manner, we get

$$\gamma^5 \equiv \frac{q^5}{[[2]]_{q^2} [[3]]_{q^2}} \varepsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (102)$$

where a normalization factor has been introduced to give the above expression for  $\gamma^5$  its simple form. In analogy to the undeformed case  $\gamma^5$  anticommutes with all  $\gamma$ -matrices:

$$\{\gamma^5, \gamma^\mu\} = 0, \quad \mu \in \{+, 3, 0, -\}. \quad (103)$$

**Definition of 'inverse'  $\gamma$ -matrices:** In part I it was stressed that in the case of  $q$ -deformed Minkowski space we have to distinguish between Pauli matrices and their so-called 'inverse' counterparts. This observation can be used to introduce 'inverse'  $\gamma$ -matrices:

$$(\gamma_\mu^{-1})_a{}^b \equiv \begin{pmatrix} 0 & (\sigma_\mu^{-1})_\alpha{}^\beta \\ (\bar{\sigma}_\mu^{-1})_{\dot{\alpha}}{}^{\dot{\beta}} & 0 \end{pmatrix}. \quad (104)$$

For a detailed discussion of the 'inverse' Pauli matrices  $\sigma_\mu^{-1}$  and  $\bar{\sigma}_\mu^{-1}$  we refer the reader to part I. Here, let us recall that the 'inverse' Pauli matrices  $\sigma_\mu^{-1}$  and  $\bar{\sigma}_\mu^{-1}$  are not inverse in the sense of matrix multiplication and the same holds for 'inverse'  $\gamma$ -matrices. Finally, it should be mentioned that bispinor indices of 'inverse'  $\gamma$ -matrices are raised according to the rules

$$(\gamma_\mu^{-1})^{ab} = (D_L)^{aa'} (\gamma_\mu^{-1})_{a'}{}^b, \quad (\gamma_\mu^{-1})_{ab} = (\gamma_\mu^{-1})_a{}^{b'} (D_R^T)_{b'b}. \quad (105)$$

These conventions follow directly from the rules for raising and lowering spinor indices of 'inverse' Pauli matrices.

To underpin our statement that 'inverse'  $\gamma$ -matrices are not inverse in the usual sense we give the expressions:

$$\begin{aligned}
(\gamma_+^{-1})_a{}^b &= \begin{pmatrix} 0 & 0 & 0 & q^{-3/2} \\ 0 & 0 & 0 & 0 \\ 0 & -q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (\gamma_-^{-1})_a{}^b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1/2} & 0 \\ 0 & 0 & 0 & 0 \\ q^{3/2} & 0 & 0 & 0 \end{pmatrix}, \\
(\gamma_3^{-1})_a{}^b &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \end{pmatrix}, (\gamma_0^{-1})_a{}^b = \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{106}$$

Nothing prevents us from introducing a matrix  $\gamma_5^{-1}$ . In analogy to (102) we have

$$\gamma_5^{-1} \equiv -\frac{q^5}{[[2]]_{q^2}[[3]]_{q^2}} \varepsilon^{\mu\nu\rho\sigma} \gamma_\mu^{-1} \gamma_\nu^{-1} \gamma_\rho^{-1} \gamma_\sigma^{-1} = \gamma^5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{107}$$

**Left- and right-dual bases:** Next, we are seeking a basis for the Clifford algebra of  $q$ -deformed Minkowski space. As in the undeformed case, such a basis can be build up from  $\gamma$ -matrices. One possibility to achieve this is given by the following set of matrices:

$$\begin{aligned}
&1, \gamma^+, \gamma^3, \gamma^0, \gamma^-, \gamma^{[+}\gamma^3]_q, \gamma^{[+}\gamma^0]_q, \gamma^{[+}\gamma^-]_q, \gamma^{[3}\gamma^0]_q, \gamma^{[3}\gamma^-]_q, \\
&\gamma^{[0}\gamma^-]_q, \gamma^{[+}\gamma^3\gamma^0]_q, \gamma^{[+}\gamma^3\gamma^-]_q, \gamma^{[+}\gamma^0\gamma^-]_q, \gamma^{[3}\gamma^0\gamma^-]_q, \gamma^{[+}\gamma^3\gamma^0\gamma^-]_q.
\end{aligned} \tag{108}$$

In the order of their appearance the matrices in (108) are denoted by  $\Gamma^e$ ,  $\Gamma^+, \dots, \Gamma^{+30-}$ . The antisymmetrized products of two, three, or four  $\gamma$ -matrices are calculated from the expressions:

$$\begin{aligned}
\gamma^{[\mu}\gamma^{\nu]}_q &\equiv (P_A)^{\mu\nu}{}_{\mu'\nu'} \gamma^{\mu'} \gamma^{\nu'}, \\
\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}_q &\equiv \frac{q^6}{2[[3]]_{q^2}} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu'\nu'\rho'\sigma} \gamma^{\mu'} \gamma^{\nu'} \gamma^{\rho'} \gamma^{\sigma'}, \\
\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}_q &\equiv -2q^{-8}[[2]]_{q^2}^2[[3]]_{q^2} \varepsilon^{+30-} \varepsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu \\
&= -2q^{-13}[[2]]_{q^2}^3[[3]]_{q^2}^2 \gamma^5.
\end{aligned} \tag{109}$$

Clearly, the antisymmetrizer  $P_A$  and the  $q$ -deformed epsilon tensor now refer to  $q$ -deformed Minkowski space.

The quantum trace for Pauli matrices (for its definition see part I) can be generalized to  $\gamma$ -matrices in the following manner:

$$\mathrm{Tr}_q \gamma^\mu \gamma^\nu \equiv \mathrm{Tr} D_L^T \gamma^\mu \gamma^\nu D_R. \quad (110)$$

This quantum trace enables us to introduce a left-dual basis of (108). Its elements  $\Gamma_{l,A}$ ,  $A \in \{e, +, \dots, +30-\}$ , are determined by

$$\mathrm{Tr}_q (\Gamma_{l,A} \Gamma^B) = \delta^B_A. \quad (111)$$

Exploiting these conditions we found

$$\begin{aligned} \Gamma_{l,e} &= -\frac{1}{2} \lambda_+^{-1} \Gamma^e, & \Gamma_{l,+} &= -\frac{q}{2} \Gamma^-, & \Gamma_{l,3} &= \frac{1}{2} \Gamma^3, \\ \Gamma_{l,0} &= -\frac{1}{2} \Gamma^0, & \Gamma_{l,-} &= -\frac{1}{2q} \Gamma^+, \end{aligned} \quad (112)$$

$$\begin{aligned} \Gamma_{l,+3} &= 2q^{-1} \lambda_+^{-1} \Gamma^{3-} + \lambda \lambda_+^{-1} \Gamma^{0-}, \\ \Gamma_{l,+0} &= -2q \lambda_+^{-1} \Gamma^{0-} + \lambda \lambda_+^{-1} \Gamma^{3-}, \\ \Gamma_{l,+-} &= -(q^2 + q^{-2}) \lambda_+^{-1} \Gamma^{+-} + \lambda \lambda_+^{-1} \Gamma^{30}, \\ \Gamma_{l,30} &= -2 \lambda_+^{-1} \Gamma^{30} + \lambda \lambda_+^{-1} \Gamma^{+-}, \\ \Gamma_{l,3-} &= 2q^{-3} \lambda_+^{-1} \Gamma^{+3} + q^{-2} \lambda \lambda_+^{-1} \Gamma^{+0}, \\ \Gamma_{l,0-} &= -2q^{-1} \lambda_+^{-1} \Gamma^{+0} + q^{-2} \lambda \lambda_+^{-1} \Gamma^{+3}, \end{aligned} \quad (113)$$

$$\begin{aligned} \Gamma_{l,+30} &= -2q^{-1} \Gamma^{30-}, & \Gamma_{l,+3-} &= 2q^{-4} \Gamma^{+3-}, & \Gamma_{l,+0-} &= -\frac{1}{2} \Gamma^{+0-}, \\ \Gamma_{l,30-} &= -2q^{-3} \Gamma^{+30}, & \Gamma_{l,+30-} &= -\frac{1}{8} q^{27} [[2]]_{q^2}^{-7} [[3]]_{q^2}^{-4} \Gamma^{+30-}. \end{aligned} \quad (114)$$

There is also the notion of a right-dual basis of (108). Its elements  $\Gamma_{r,A}$ ,  $A \in \{e, +, \dots, +30-\}$  are subject to

$$\mathrm{Tr}_q (\Gamma^A \Gamma_{r,B}) = \delta^A_B. \quad (115)$$

The elements of the right-dual basis are obtained from those of the left-dual one. Concretely, we have the correspondences

$$\Gamma_{r,+} = q^{-2} \Gamma_{l,+}, \quad \Gamma_{r,-} = q^2 \Gamma_{l,-},$$

$$\begin{aligned}\Gamma_{r,+3} &= q^{-2} \Gamma_{l,+3}, & \Gamma_{r,+0} &= q^{-2} \Gamma_{l,+0}, & \Gamma_{r,3-} &= q^2 \Gamma_{l,3-}, & \Gamma_{r,0-} &= q^2 \Gamma_{l,0-}, \\ \Gamma_{r,+30} &= q^{-2} \Gamma_{l,+30}, & \Gamma_{r,30-} &= q^2 \Gamma_{l,30-}.\end{aligned}\quad (116)$$

For the remaining matrices it holds  $\Gamma_{r,A} = \Gamma_{l,A}$ .

Now, we can repeat the same reasonings as above for 'inverse' matrices. Let us recall that the 'inverse'  $\gamma$ -matrices are representations of the elements  $\tilde{\xi}_\mu$ ,  $\mu \in \{+, 0, 3, -\}$ . In this sense, they fulfill the relations in (37). The 16 matrices

$$\begin{aligned}1, & \gamma_\mu^{-1}, \mu \in \{+, 3, 0, -\}, \\ \gamma_{[\mu}^{-1} \gamma_{\nu]}^{-1}, & (\mu, \nu) \in \{(+, 3), (+, 0), (+, -), (3, -), (3, 0), (0, -)\}, \\ \gamma_{[\mu}^{-1} \gamma_{\nu}^{-1} \gamma_{\rho]}^{-1}, & (\mu, \nu, \rho) \in \{(+, 3, -), (+, 0, -), (3, 0, -)\}, \\ \gamma_{[+}^{-1} \gamma_3^{-1} \gamma_0^{-1} \gamma_{-]}^{-1}, & \end{aligned}\quad (117)$$

where

$$\begin{aligned}\gamma_{[\mu}^{-1} \gamma_{\nu]}^{-1} &\equiv (P_A)^{\mu'\nu'}{}_{\mu\nu} \gamma_{\mu'}^{-1} \gamma_{\nu'}^{-1}, \\ \gamma_{[\mu}^{-1} \gamma_{\nu}^{-1} \gamma_{\rho]}^{-1} &\equiv \frac{q^6}{2[[3]]_{q^2}} \varepsilon_{\sigma\rho\nu\mu} \varepsilon^{\sigma\rho'\nu'\mu'} \gamma_{\rho'}^{-1} \gamma_{\nu'}^{-1} \gamma_{\mu'}^{-1}, \\ \gamma_{[\mu}^{-1} \gamma_{\nu}^{-1} \gamma_{\rho}^{-1} \gamma_{\sigma]}^{-1} &\equiv -2q^{-8} [[2]]_{q^2}^2 [[3]]_{q^2} \varepsilon_{-03+} \varepsilon^{\sigma\rho\nu\mu} \gamma_{\sigma}^{-1} \gamma_{\rho}^{-1} \gamma_{\nu}^{-1} \gamma_{\mu}^{-1} \\ &= 2q^{-13} [[2]]_{q^2}^3 [[3]]_{q^2}^2 \gamma_5^{-1},\end{aligned}\quad (118)$$

establish a basis of the Clifford algebra subject to the relations in (37). We refer to the matrices in (117) as  $\Gamma_A^{-1}$ ,  $A \in \{1, +, 3, 0, \dots, +30 - \}$ .

To determine a dual basis of (117) we first have to adjust the definition of the quantum trace to 'inverse'  $\gamma$ -matrices. As a consequence of the quantum trace for 'inverse' Pauli matrices [cf. part I] this task can be achieved by

$$\text{Tr}_q \gamma_\mu^{-1} \gamma_\nu^{-1} \equiv \text{Tr } D_L \gamma_\mu^{-1} \gamma_\nu^{-1} D_R^T. \quad (119)$$

We require that the matrices  $(\Gamma^{-1})^{l,A}$  of a left-dual basis satisfy

$$\text{Tr}_q (\Gamma^{-1})^A \Gamma_B^{-1} = \delta^A_B. \quad (120)$$

We simply get the matrices  $(\Gamma^{-1})^{l,A}$ ,  $A \in \{1, +, 3, 0, \dots, +30 - \}$ , by applying the replacements

$$\Gamma^A \rightarrow \Gamma_A^{-1}, \quad \Gamma_{l,A} \rightarrow (\Gamma^{-1})^{l,A}, \quad (121)$$

together with

$$\lambda \rightarrow -\lambda, \quad (122)$$

to the formulae in (112)-(114). This procedure, for example, yields

$$(\Gamma^{-1})^{l,+ -} = -q^{-2}(q^4 + 1)\lambda_+^{-1}\Gamma_{+-}^{-1} - \lambda\lambda_+^{-1}\Gamma_{30}^{-1}. \quad (123)$$

**Completeness relations:** The  $\Gamma^A$  constitute a basis in the space of  $4 \times 4$  matrices, so we are able to expand an arbitrary matrix  $M$  as

$$M = \sum_A c_A \Gamma^A. \quad (124)$$

The coefficients are obtained from  $M$  by the trace formula

$$c_B = \text{Tr}_q(M \Gamma_{r,B}) = \text{Tr}_q(\Gamma_{l,B} M), \quad (125)$$

which is a direct consequence of (111) and (115).

We can use this trace expansion to deduce the completeness relation

$$\begin{aligned} \delta_a^b \delta_c^d &= \sum_B (D_R)_{cm} (D_L^T)^{mk} (\Gamma_{l,B})_k^b (\Gamma^B)_a^d \\ &= \sum_B (\Gamma^B)_a^d (\Gamma_{r,B})_c^k (D_R)_{km} (D_L^T)^{mb}, \end{aligned} \quad (126)$$

Introducing the new matrices

$$\begin{aligned} (\Gamma_{l,B}^c)_a^b &\equiv (D_R)_{am} (D_L^T)^{mk} (\Gamma_{l,B})_k^b, \\ (\Gamma_{l,B}^c)_a^b &\equiv (\Gamma_{r,B})_a^k (D_R)_{km} (D_L^T)^{mb}. \end{aligned} \quad (127)$$

we finally arrive at the relations

$$\sum_B (\Gamma_{l,B}^c)_c^b (\Gamma^B)_a^d = \delta_a^b \delta_c^d, \quad \sum_B (\Gamma^B)_a^d (\Gamma_{r,B}^c)_c^b = \delta_a^b \delta_c^d. \quad (128)$$

Incidentally, the matrices  $\Gamma_B^c$  are related to the  $\Gamma^B$ ,  $B \in \{1, +, 3, 0, \dots, +30 -\}$ , via the identities

$$\begin{aligned} \Gamma_1^c &= \frac{1}{2}q^{-2}(q^4 + 1)\lambda_+^{-2}\Gamma^e + \frac{1}{2}\lambda\lambda_+^{-1}\Gamma^{+-}, \quad \Gamma_+^c = \frac{1}{2}\Gamma^-, \\ \Gamma_3^c &= -\lambda_+^{-1}\Gamma^3 - \frac{1}{4}\lambda^2\lambda_+^{-1}\Gamma^0 + \frac{1}{2}q^{-2}\lambda\Gamma^{+30}, \end{aligned}$$

$$\begin{aligned}\Gamma_0^c &= \frac{1}{2}q^{-2}(q^4+1)\lambda_+^{-1}\Gamma^0 - \frac{1}{4}\lambda^2\lambda_+^{-1}\Gamma^3 + \frac{1}{2}q^{-1}\lambda^2\Gamma^{+30} - \frac{1}{2}\lambda\Gamma^{+0-}, \\ \Gamma_-^c &= \frac{1}{2}\Gamma^+, \quad \Gamma_{+3}^c = -2q^{-2}\lambda_+^{-1}\Gamma^{3-} - q^{-1}\lambda\lambda_+^{-1}\Gamma^{0-},\end{aligned}\tag{129}$$

$$\begin{aligned}\Gamma_{+0}^c &= 2\lambda_+^{-1}\Gamma^{0-} - q^{-1}\lambda\lambda_+^{-1}\Gamma^{3-}, \\ \Gamma_{+-}^c &= 2q^{-2}(q^4+1)\lambda_+^{-2}\Gamma^{+-} - 2\lambda\lambda_+^{-2}\Gamma^{+30-} \\ &\quad - \frac{1}{4}q^{10}\lambda^2\lambda_+^{-5}[[3]]_{q^2}^{-2}\Gamma^{+30-} + \frac{1}{2}\lambda\lambda_+^{-1}\Gamma^e, \\ \Gamma_{30}^c &= 4\lambda_+^{-2}\Gamma^{30} - 2\lambda\lambda_+^{-2}\Gamma^{+-} + \frac{1}{2}q^{11}\lambda^2\lambda_+^{-5}[[3]]_{q^2}^{-2}\Gamma^{+30-}, \\ \Gamma_{3-}^c &= -2q^{-2}\lambda_+^{-1}\Gamma^{+3} - q^{-1}\lambda\lambda_+^{-1}\Gamma^{0-}, \\ \Gamma_{0-}^c &= 2\lambda_+^{-1}\Gamma^{+0} - q^{-1}\lambda\lambda_+^{-1}\Gamma^{+3}, \quad \Gamma_{+30}^c = 2q^{-2}\Gamma^{30-},\end{aligned}\tag{130}$$

$$\begin{aligned}\Gamma_{+3-}^c &= 2q^{-2}\lambda_+^{-1}(\lambda\lambda_+ - 2)\Gamma^{+3-} - \frac{1}{2}q^{-2}\lambda\Gamma^3 + \frac{1}{2}q^{-1}\lambda^2\Gamma^0 \\ &\quad - q^{-3}(3q^2+1)\lambda\lambda_+^{-1}\Gamma^{+0-}, \\ \Gamma_{+0-}^c &= 4\lambda_+^{-1}\Gamma^{+0-} - q^{-3}(3q^2+1)\lambda\lambda_+^{-1}\Gamma^{+3-} - \frac{1}{2}\Gamma^0, \\ \Gamma_{30-}^c &= 2q^{-2}\Gamma^{+30},\end{aligned}\tag{131}$$

$$\begin{aligned}\Gamma_{+30-}^c &= \frac{1}{8}q^{18}(q^4+1)\lambda_+^{-8}[[3]]_{q^2}^{-4}\Gamma^{+30-} - \frac{1}{4}q^{10}\lambda^2\lambda_+^{-5}[[3]]_{q^2}^{-2}\Gamma^{+-} \\ &\quad + \frac{1}{2}q^{11}\lambda^2\lambda_+^{-5}[[3]]_{q^2}^{-2}\Gamma^{30}.\end{aligned}\tag{132}$$

It is not very difficult to adjust the above reasonings to 'inverse'  $\gamma$ -matrices. The set of  $\Gamma_A^{-1}$ ,  $A \in \{1, +, 3, 0, \dots, +30 -\}$ , again give us a basis of  $4 \times 4$  matrices:

$$M = \sum_A c^A \Gamma_A^{-1},\tag{133}$$

where

$$c^B = \text{Tr}_q(M (\Gamma^{-1})^{r,B}) = \text{Tr}_q((\Gamma^{-1})^{l,B} M).\tag{134}$$

These identities imply

$$\begin{aligned}\delta_a^b \delta_c^d &= \sum_B (D_R^T)_{cm} (D_L)^{mk} ((\Gamma^{-1})^{l,B})_k^b ((\Gamma_B^{-1})_a^d \\ &= \sum_B (\Gamma_B^{-1})_a^d ((\Gamma^{-1})^{r,B})_c^k (D_R)_{km} (D_L^T)^{mb},\end{aligned}\tag{135}$$

or, more compactly,

$$\begin{aligned}\sum_B ((\Gamma_c^{-1})^{l,B})_c^b (\Gamma_B^{-1})_a^d &= \delta_a^b \delta_c^d, \\ \sum_B (\Gamma_B^{-1})_a^d ((\Gamma_c^{-1})^{l,B})_c^b &= \delta_a^b \delta_c^d,\end{aligned}\tag{136}$$

with

$$\begin{aligned}((\Gamma_c^{-1})^{l,B})_a^b &\equiv (D_R^T)_{am} (D_L)^{mk} ((\Gamma^{-1})^{l,B})_k^b, \\ ((\Gamma_c^{-1})^{l,B})_a^b &\equiv ((\Gamma^{-1})^{r,B})_a^k (D_R^T)_{km} (D_L)^{mb}.\end{aligned}\tag{137}$$

Once again, we can express the matrices  $(\Gamma_c^{-1})^A$  by the  $\Gamma_A^{-1}$ ,  $A \in \{1, +, -, 3, 0, \dots, +30, -\}$ :

$$\begin{aligned}(\Gamma_c^{-1})^e &= \frac{1}{2} q^{-2} (q^4 + 1) \lambda_+^{-2} \Gamma_e^{-1} + \frac{1}{2} \lambda \lambda_+^{-1} \Gamma_{+-}^{-1}, \quad (\Gamma_c^{-1})^+ = \frac{1}{2} \Gamma_-^{-1}, \\ (\Gamma_c^{-1})^3 &= -\lambda_+^{-1} \Gamma_3^{-1} - \frac{1}{4} \lambda^2 \lambda_+^{-1} \Gamma_0^{-1} + \frac{1}{2} q^{-2} \lambda \Gamma_{+30}^{-1}, \\ (\Gamma_c^{-1})^0 &= \frac{1}{2} q^{-2} (q^4 + 1) \lambda_+^{-1} \Gamma_0^{-1} + \frac{1}{4} \lambda^2 \lambda_+^{-1} \Gamma_3^{-1} - \frac{1}{2} q^{-1} \lambda^2 \Gamma_{+3-}^{-1} - \frac{1}{2} \lambda \Gamma_{+0-}^{-1}, \\ (\Gamma_c^{-1})^- &= \frac{1}{2} \Gamma_+^{-1}, \quad (\Gamma_c^{-1})^{+3} = -2q^{-2} \lambda_+^{-1} \Gamma_{3-}^{-1} + q^{-1} \lambda \lambda_+^{-1} \Gamma_{0-}^{-1},\end{aligned}\tag{138}$$

$$\begin{aligned}(\Gamma_c^{-1})^{+0} &= 2\lambda_+^{-1} \Gamma_{0-}^{-1} + q^{-1} \lambda \lambda_+^{-1} \Gamma_{3-}^{-1}, \\ (\Gamma_c^{-1})^{+-} &= 2q^{-2} (q^4 + 1) \lambda_+^{-2} \Gamma_{+-}^{-1} + 2\lambda \lambda_+^{-2} \Gamma_{30}^{-1} \\ &\quad + \frac{1}{4} q^{10} \lambda^2 \lambda_+^{-5} [[3]]_{q^2}^{-2} \Gamma_{+30-}^{-1} + \frac{1}{2} \lambda \lambda_+^{-1} \Gamma_e^{-1}, \\ (\Gamma_c^{-1})^{30} &= 4\lambda_+^{-2} \Gamma_{30}^{-1} + 2\lambda \lambda_+^{-2} \Gamma_{+-}^{-1} + \frac{1}{2} q^{11} \lambda^2 \lambda_+^{-5} [[3]]_{q^2}^{-2} \Gamma_{+30-}^{-1}, \\ (\Gamma_c^{-1})^{3-} &= -2q^{-2} \lambda_+^{-1} \Gamma_{+3}^{-1} + q^{-1} \lambda \lambda_+^{-1} \Gamma_{+0}^{-1}, \\ (\Gamma_c^{-1})^{0-} &= 2\lambda_+^{-1} \Gamma_{+0}^{-1} + q^{-1} \lambda \lambda_+^{-1} \Gamma_{+3}^{-1}, \quad (\Gamma_c^{-1})^{+30} = 2q^{-2} \Gamma_{30-}^{-1},\end{aligned}\tag{139}$$

$$\begin{aligned}(\Gamma_c^{-1})^{+3-} &= 2q^{-2} \lambda_+^{-1} (\lambda \lambda_+ - 2) \Gamma_{+3-}^{-1} - \frac{1}{2} q^{-2} \lambda \Gamma_3^{-1} - \frac{1}{2} q^{-1} \lambda^2 \Gamma_0^{-1} \\ &\quad + q^{-3} (3q^2 + 1) \lambda \lambda_+^{-1} \Gamma_{+0-}^{-1}, \\ (\Gamma_c^{-1})^{+0-} &= 4\lambda_+^{-1} \Gamma_{+0-}^{-1} - q^{-3} (3q^2 + 1) \lambda \lambda_+^{-1} \Gamma_{+3-}^{-1} - \frac{1}{2} \Gamma_0^{-1}, \\ (\Gamma_c^{-1})^{30-} &= 2q^{-2} \Gamma_{+30}^{-1},\end{aligned}$$

$$\begin{aligned}
(\Gamma_c^{-1})^{+30-} &= \frac{1}{8}q^{18}(q^4+1)\lambda_+^{-8}[[3]]_{q^2}^{-4}\Gamma_{+30-}^{-1} + \frac{1}{4}q^{10}\lambda^2\lambda_+^{-5}[[3]]_{q^2}^{-2}\Gamma_{+-}^{-1} \\
&\quad + \frac{1}{2}q^{11}\lambda^2\lambda_+^{-5}[[3]]_{q^2}^{-2}\Gamma_{30}^{-1}.
\end{aligned} \tag{140}$$

**A more physical oriented basis:** In physical applications it is more convenient to work with a basis that uses the matrices  $\Gamma^5 \equiv \gamma^5$  and  $\Gamma^{5\mu} \equiv \gamma^5\gamma^\mu$ ,  $\mu \in \{+, 3, 0, -\}$ , instead of antisymmetrized products of three and four  $\gamma$ -matrices:

$$\begin{aligned}
&1, \gamma^+, \gamma^3, \gamma^0, \gamma^-, \gamma^5, \gamma^5\gamma^+, \gamma^5\gamma^3, \gamma^5\gamma^0, \gamma^5\gamma^-, \\
&\gamma^{[+\gamma^3]_q}, \gamma^{[+\gamma^0]_q}, \gamma^{[+\gamma^-]_q}, \gamma^{[3\gamma^0]_q}, \gamma^{[3\gamma^-]_q}, \gamma^{[0\gamma^-]_q}.
\end{aligned} \tag{141}$$

From (111) we find that the left-duals of  $\Gamma^5$  and  $\Gamma^{5\mu}$  become

$$\begin{aligned}
\Gamma_{l,5+} &= \frac{q}{2}\Gamma^{5-}, \quad \Gamma_{l,53} = -\frac{1}{2}\Gamma^{53}, \\
\Gamma_{l,50} &= \frac{1}{2}\Gamma^{50}, \quad \Gamma_{l,5-} = \frac{1}{2}q^{-1}\Gamma^{5+}, \quad \Gamma_{l,5} = \frac{1}{2}\lambda_+^{-1}\Gamma^5.
\end{aligned} \tag{142}$$

Furthermore, if we work with the basis of (141) the completeness relations in (128) require to deal with the matrices

$$\begin{aligned}
\Gamma_3^c &= -\lambda_+^{-1}\Gamma^3 - \frac{1}{4}\lambda^2\lambda_+^{-1}\Gamma^0 + \frac{1}{4}\lambda\Gamma^{50}, \\
\Gamma_0^c &= \frac{1}{2}q^{-2}(q^4+1)\lambda_+^{-1}\Gamma^0 - \frac{1}{4}\lambda^2\lambda_+^{-1}\Gamma^3 - \frac{1}{4}\lambda\Gamma^{53}, \\
\Gamma_5^c &= \frac{1}{2}q^{-2}(q^4+1)\lambda_+^{-2}\Gamma^5 + \frac{1}{2}\lambda^2\lambda_+^2\Gamma^{+-} - \lambda\lambda_+^{-1}\Gamma^{30},
\end{aligned} \tag{143}$$

$$\begin{aligned}
\Gamma_{5+}^c &= -\frac{1}{2}\Gamma^{5-}, \\
\Gamma_{53}^c &= \lambda_+^{-1}\Gamma^{53} - \frac{1}{4}\lambda\Gamma^0 + \frac{1}{4}\lambda^2\lambda_+^{-1}\Gamma^{50}, \\
\Gamma_{50}^c &= -\frac{1}{2}q^{-2}(q^4+1)\lambda_+^{-1}\Gamma^{50} + \frac{1}{4}\lambda\Gamma^3 + \frac{1}{4}\lambda^2\lambda_+^{-1}\Gamma^{53}, \\
\Gamma_{5-}^c &= -\frac{1}{2}\Gamma^{5+}.
\end{aligned} \tag{144}$$

Replacing in (141) all  $\gamma$ -matrices by their 'inverse' counterparts we are led to a basis of the Clifford algebra subject to the relations in (37):

$$1, \gamma^5, \gamma_\mu^{-1}, \gamma_5^{-1}\gamma_\mu^{-1}, \mu \in \{+, 3, 0, -\},$$



$$\gamma_{[\mu}^{-1} \gamma_{\nu]}^{-1}, \quad (\mu, \nu) \in \{(+, 3), (+, 0), (+, -), (3, -), (3, 0), (0, -)\}. \quad (145)$$

Applying the substitutions  $\Gamma^A \rightarrow \Gamma_A^{-1}$ ,  $\Gamma_{l,A} \rightarrow (\Gamma^{-1})^{l,A}$  to (142) gives us the expressions for the left-duals  $\Gamma_{l,5}^{-1}$  and  $\Gamma_{l,5\mu}^{-1}$ ,  $\mu \in \{+, 3, 0, -\}$ . If we want to work with the matrices  $(\Gamma_c^{-1})^A$  corresponding to the basis (145) we also need

$$\begin{aligned} (\Gamma_c^{-1})^3 &= -\lambda_+^{-1} \Gamma_3^{-1} + \frac{1}{4} \lambda^2 \lambda_+^{-1} \Gamma_0^{-1} - \frac{1}{4} \lambda \Gamma_{50}^{-1}, \\ (\Gamma_c^{-1})^0 &= \frac{1}{2} q^{-2} (q^4 + 1) \lambda_+^{-1} \Gamma_0^{-1} + \frac{1}{4} \lambda^2 \lambda_+^{-1} \Gamma_3^{-1} + \frac{1}{4} \lambda \Gamma_{53}^{-1}, \\ (\Gamma_c^{-1})^5 &= \frac{1}{2} q^{-2} (q^4 + 1) \lambda_+^{-2} \Gamma_5^{-1} + \frac{1}{2} \lambda^2 \lambda_+^2 \Gamma_{+-}^{-1} + \lambda \lambda_+^{-1} \Gamma_{30}^{-1}, \end{aligned} \quad (146)$$

$$\begin{aligned} (\Gamma_c^{-1})^{5+} &= -\frac{1}{2} \Gamma_{5-}^{-1}, \\ (\Gamma_c^{-1})^{53} &= \lambda_+^{-1} \Gamma_{53}^{-1} + \frac{1}{4} \lambda \Gamma_0^{-1} - \frac{1}{4} \lambda^2 \lambda_+^{-1} \Gamma_{50}^{-1}, \\ (\Gamma_c^{-1})^{50} &= -\frac{1}{2} q^{-2} (q^4 + 1) \lambda_+^{-1} \Gamma_{50}^{-1} - \frac{1}{4} \lambda \Gamma_3^{-1} - \frac{1}{4} \lambda^2 \lambda_+^{-1} \Gamma_{53}^{-1}, \\ (\Gamma_c^{-1})^{5-} &= -\frac{1}{2} \Gamma_{5+}^{-1}. \end{aligned} \quad (147)$$

### 3.2.2 Relations concerning $q$ -deformed Dirac matrices

In the literature one can find numerous identities for  $\gamma$ -matrices (see, for example, Refs. [44–49]). Many of them play an important role in quantum field theory. This subsection is devoted to  $q$ -analogs of such relations. It should be emphasized that the objects ( $\hat{R}$ -matrix, projectors, quantum metric, epsilon tensor,  $\gamma$ -matrices etc.) appearing in the following formulae all refer to  $q$ -deformed Minkowski space and the matrix product of  $\gamma$ -matrices is understood as  $\gamma^\mu \gamma^\nu = (\gamma^\mu)_a{}^{a'} (\gamma^\nu)_{a'}{}^b$ .

We begin with trace relations for  $\gamma$ -matrices:

$$\begin{aligned} \text{Tr}_q \gamma^\mu &= 0, \quad \text{Tr}_q \gamma^\mu \gamma^\nu = 2\eta^{\mu\nu}, \quad \text{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho = 0, \\ \text{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma &= -q^{-1} \eta^{\mu\nu} \eta^{\rho\sigma} + q \eta^{\mu\kappa} \hat{R}^{\nu\rho}{}_{\kappa\lambda} \eta^{\lambda\sigma} - q \eta^{\mu\sigma} \eta^{\nu\rho} \\ &= -q \eta^{\mu\nu} \eta^{\rho\sigma} + q^{-1} \eta^{\mu\kappa} (\hat{R}^{-1})^{\nu\rho}{}_{\kappa\lambda} \eta^{\lambda\sigma} - q^{-1} \eta^{\mu\sigma} \eta^{\nu\rho}. \end{aligned} \quad (148)$$

In addition to this, we could also check the validity of the following trace formula:

$$\text{Tr}_q \gamma^5 \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} \gamma^{\mu_6} =$$

$$\begin{aligned}
&= -2q^{-1} \left[ k_1 \eta^{\mu_1 \mu_2} \varepsilon^{\mu_3 \mu_4 \mu_5 \mu_6} + k_2 \eta^{\mu_1 \mu'_3} \hat{R}^{\mu_2 \mu_3}{}_{\mu'_3 \mu'_2} \varepsilon^{\mu'_2 \mu_4 \mu_5 \mu_6} \right. \\
&\quad + k_3 \eta^{\mu_2 \mu_3} \varepsilon^{\mu_1 \mu_4 \mu_5 \mu_6} + k_4 \eta^{\mu_4 \mu_5} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_6} \\
&\quad \left. + k_5 \eta^{\mu_4 \mu'_6} \hat{R}^{\mu_5 \mu_6}{}_{\mu'_6 \mu'_5} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu'_5} + k_6 \eta^{\mu_5 \mu_6} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \right], \quad (149)
\end{aligned}$$

where the values of the coefficients are

$$k_1 = 1, \quad k_2 = -q^{-1}, \quad k_3 = q^{-2}, \quad k_4 = 1, \quad k_5 = -q^{-1}, \quad k_6 = q^{-2}. \quad (150)$$

Notice that in (149) we are allowed to replace  $\hat{R}$  with  $\hat{R}^{-1}$ , if we instead set

$$k_1 = q^{-2}, \quad k_2 = -q^{-1}, \quad k_3 = 1, \quad k_4 = q^{-2}, \quad k_5 = -q^{-1}, \quad k_6 = 1. \quad (151)$$

Finally, we could also calculate trace formulae with  $\gamma^5$ :

$$\text{Tr}_q \gamma^5 = \text{Tr}_q \gamma^\mu \gamma^5 = \text{Tr}_q \gamma^\mu \gamma^\nu \gamma^5 = \text{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^5 = 0, \quad (152)$$

and

$$\text{Tr}_q \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = q^2 \varepsilon^{\mu\nu\rho\sigma}. \quad (153)$$

Next, we turn our attention to relations in which two  $\gamma$ -matrices are contracted by the quantum metric (possibly via  $\hat{R}$ -matrices). We found

$$\begin{aligned}
&\eta_{\mu\nu} \gamma^\mu \gamma^\nu = -\lambda_+ \\
&q^4 [[3]]_q^{-1} \varepsilon_{\lambda\rho\nu\kappa} \eta^{\kappa\mu} \gamma^\nu \gamma^\rho \gamma^\lambda = \gamma^5 \gamma^\nu, \\
&\hat{R}^{\mu\nu}{}_{\nu'\mu'} \eta_{\rho\mu} \gamma^\rho \gamma^{\nu'} \gamma^{\mu'} = q^{-1} \gamma^\nu, \\
&\hat{R}^{\mu\nu}{}_{\nu'\mu'} \hat{R}^{\mu'\rho}{}_{\rho'\mu''} \eta_{\lambda\mu} \gamma^\lambda \gamma^{\nu'} \gamma^{\rho'} \gamma^{\mu''} = q^{-2} \eta^{\nu\rho}, \quad (154)
\end{aligned}$$

and the more lengthy relation

$$\begin{aligned}
&\hat{R}^{\mu\nu}{}_{\nu'\mu'} \hat{R}^{\mu'\rho}{}_{\rho'\mu''} \hat{R}^{\mu''\sigma}{}_{\sigma'\mu'''} \eta_{\lambda\mu} \gamma^\lambda \gamma^{\nu'} \gamma^{\rho'} \gamma^{\sigma'} \gamma^{\mu'''} \\
&= q^{-3} \hat{R}^{\rho''\sigma''}{}_{\sigma'\rho'} \hat{R}^{\nu''\sigma}{}_{\sigma''\nu'} \hat{R}^{\nu\rho}{}_{\rho''\nu''} \gamma^{\sigma'} \gamma^{\rho'} \gamma^{\nu'}, \quad (155)
\end{aligned}$$

which can be viewed as  $q$ -analog of the undeformed relation  $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$  [44]. Let us note that we get variants of the relations in (154) and (155) through the replacements  $\hat{R} \rightarrow \hat{R}^{-1}$ ,  $q \rightarrow q^{-1}$ .

A product of three  $\gamma$ -matrices can be decomposed as follows:

$$\gamma^\mu \gamma^\nu \gamma^\rho = \gamma^{[\mu} \gamma^\nu \gamma^{\rho]} + \frac{1}{2} q^{-1} \eta^{\mu\nu} \gamma^\rho$$

$$\begin{aligned}
& + \frac{q}{2} \eta^{\mu\lambda'} \hat{R}^{\nu\lambda}{}_{\lambda'\nu'} \gamma^{\nu'} - \frac{q}{2} \gamma^\mu \eta^{\nu\lambda} \\
& = \gamma^{[\mu} \gamma^{\nu} \gamma^{\rho]} q - \frac{1}{2} q \eta^{\mu\nu} \gamma^\rho \\
& + \frac{1}{2} q^{-1} \eta^{\mu\lambda'} (\hat{R}^{-1})^{\nu\lambda}{}_{\lambda'\nu'} \gamma^{\nu'} - \frac{1}{2} q^{-1} \gamma^\mu \eta^{\nu\lambda}. \tag{156}
\end{aligned}$$

All relations in this subsection can alternatively be formulated in terms of 'inverse'  $\gamma$ -matrices. We simply have to make the replacements

$$\begin{aligned}
\gamma^\mu & \rightarrow \gamma_\mu^{-1}, & \gamma^5 & \rightarrow \gamma_5^{-1}, \\
\eta^{\mu\nu} & \rightarrow \eta_{\mu\nu}, & \varepsilon^{\mu\nu\rho\sigma} & \rightarrow \varepsilon_{\sigma\rho\nu\mu}, & \varepsilon_{\mu\nu\rho\sigma} & \rightarrow \varepsilon^{\sigma\rho\nu\mu}, \\
\hat{R}^{\mu\nu}{}_{\rho\sigma} & \rightarrow \hat{R}^{\rho\sigma}{}_{\mu\nu}, & (\hat{R}^{-1})^{\mu\nu}{}_{\rho\sigma} & \rightarrow (\hat{R}^{-1})^{\rho\sigma}{}_{\mu\nu}. \tag{157}
\end{aligned}$$

In doing so, we have to be aware of the fact that the quantum trace for 'inverse' matrices is given by (119).

### 3.2.3 The $q$ -deformed spin matrices and some more relations

In analogy to the definition of two-dimensional spin matrices [cf. part I] we define the four-dimensional spin matrices as antisymmetrized product of two  $\gamma$ -matrices:

$$\begin{aligned}
(\Sigma^{\mu\nu})_a{}^b & \equiv (P_A)^{\mu\nu}{}_{\mu'\nu'} (\gamma^{\mu'})_a{}^{a'} (\gamma^{\nu'})_{a'}{}^b, \\
(\Sigma_{\mu\nu}^{-1})_a{}^b & \equiv (P_A)^{\mu'\nu'}{}_{\mu\nu} (\gamma_{\mu'}^{-1})_a{}^{a'} (\gamma_{\nu'}^{-1})_{a'}{}^b. \tag{158}
\end{aligned}$$

Raising of indices works in the same manner as for the  $\gamma$ -matrices. The  $\Sigma$ -matrices can also be written in the form

$$\begin{aligned}
(\Sigma^{\mu\nu})_a{}^b & = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha{}^\beta & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix}, \\
(\Sigma_{\mu\nu}^{-1})_a{}^b & = \begin{pmatrix} (\sigma_{\mu\nu}^{-1})_\alpha{}^\beta & 0 \\ 0 & (\bar{\sigma}_{\mu\nu}^{-1})_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix}, \tag{159}
\end{aligned}$$

with  $\sigma^{\mu\nu}$ ,  $\bar{\sigma}^{\mu\nu}$ ,  $\sigma_{\mu\nu}^{-1}$ , and  $\bar{\sigma}_{\mu\nu}^{-1}$  standing for the two-dimensional spin matrices of  $q$ -deformed Minkowski space. Exploiting the properties of two-dimensional spin matrices as they were discussed in part I, one can find from (159) that

$$(\Sigma^{\mu\nu})_{ab} (\Sigma_{\rho\sigma}^{-1})^{ab} = (\Sigma_{\rho\sigma}^{-1})^{ab} (\Sigma^{\mu\nu})_{ab} = \lambda_+ (P_A)^{\mu\nu}{}_{\rho\sigma}. \tag{160}$$

Next, we come to useful decomposition formulae involving products of  $\gamma$ - and  $\Sigma$ -matrices. For their undeformed counterparts we refer to Ref. [48], for example. Inserting the expressions for  $\gamma$ - and  $\Sigma$ -matrices into suitable ansatz, we could derive the identities

$$\begin{aligned}\gamma^\mu \gamma^\nu &= \Sigma^{\mu\nu} - \lambda_+^{-1} \eta^{\mu\nu}, \\ \gamma^5 \gamma^\mu \gamma^\nu &= -\lambda_+^{-1} \eta^{\mu\nu} \gamma^5 - q^2 \lambda_+^{-1} \varepsilon^{\mu\nu\rho\sigma} \eta_{\rho\rho'} \eta_{\sigma\sigma'} \Sigma^{\rho'\sigma'},\end{aligned}\tag{161}$$

$$\begin{aligned}\gamma^5 \Sigma^{\mu\nu} &= -q^{-2} \lambda_+^{-1} \varepsilon^{\mu\nu\rho\sigma} \eta_{\rho\rho'} \eta_{\sigma\sigma'} \Sigma^{\rho'\sigma'}, \\ \gamma^\rho \Sigma^{\mu\nu} &= \frac{q^2}{2} \varepsilon^{\rho\mu\nu\lambda} \eta_{\lambda\delta} \gamma^5 \gamma^\delta - \frac{1}{2} \lambda_+^{-1} (P_A)^{\mu\nu}{}_{\mu'\nu'} \eta^{\rho\mu'} \gamma^{\nu'\rho}, \\ \Sigma^{\mu\nu} \gamma^\rho &= \frac{q^2}{2} \varepsilon^{\mu\nu\rho\lambda} \eta_{\lambda\delta} \gamma^5 \gamma^\delta - \frac{1}{2} \lambda_+^{-1} (P_A)^{\mu\nu}{}_{\mu'\nu'} \eta^{\nu'\rho} \gamma^{\mu'\rho},\end{aligned}\tag{162}$$

$$\begin{aligned}\gamma^5 \gamma^\rho \Sigma^{\mu\nu} &= \frac{q^2}{2} \varepsilon^{\rho\mu\nu\lambda} \eta_{\lambda\delta} \gamma^\delta - \frac{1}{2} \lambda_+^{-1} (P_A)^{\mu\nu}{}_{\mu'\nu'} \eta^{\rho\mu'} \gamma^5 \gamma^{\nu'\rho}, \\ \Sigma^{\mu\nu} \gamma^5 \gamma^\rho &= \frac{q^2}{2} \varepsilon^{\mu\nu\rho\lambda} \eta_{\lambda\delta} \gamma^\delta - \frac{1}{2} \lambda_+^{-1} (P_A)^{\mu\nu}{}_{\mu'\nu'} \eta^{\nu'\rho} \gamma^5 \gamma^{\mu'\rho},\end{aligned}\tag{163}$$

$$\begin{aligned}\Sigma^{\mu\nu} \Sigma^{\kappa\lambda} &= \frac{1}{2} (P_A)^{\mu\nu}{}_{\mu'\nu'} (P_A)^{\kappa\lambda}{}_{\kappa'\lambda'} \eta^{\mu'\lambda'} \eta^{\nu'\kappa'} - \frac{q^2}{2} \lambda_+^{-1} \varepsilon^{\mu\nu\kappa\lambda} \gamma^5, \\ &\quad - \lambda_+ (P_A)^{\mu\nu}{}_{\mu'\nu'} (P_A)^{\kappa\lambda}{}_{\kappa'\lambda'} \eta^{\nu'\kappa'} \Sigma^{\mu'\lambda'}.\end{aligned}\tag{164}$$

Taking the quantum trace over  $\Sigma$ -matrices and their products yields

$$\begin{aligned}\text{Tr}_q \Sigma^{\mu\nu} &= 0, \\ \text{Tr}_q \Sigma^{\rho\sigma} \Sigma^{\mu\nu} &= q^{-1} \lambda_+ (P_A)^{\mu\nu}{}_{\mu'\nu'} \hat{R}^{\sigma\mu'}{}_{\mu''\sigma'} \eta^{\rho\mu''} \eta^{\sigma'\nu'},\end{aligned}\tag{165}$$

where in the last identity we are allowed to apply the substitutions  $\hat{R} \rightarrow \hat{R}^{-1}$ ,  $q \rightarrow q^{-1}$ . Notice that the first relation in (165) conforms with the observation that the two-dimensional spin matrices  $\sigma^{\mu\nu}$  and  $\bar{\sigma}^{\mu\nu}$  are traceless (in a  $q$ -deformed sense).

Now, we write down  $q$ -deformed analoga of some relations given on page 13 of Ref. [48], for example:

$$\eta_{\mu\mu'} \hat{R}^{\mu'\nu}{}_{\nu'\mu''} \hat{R}^{\mu''\rho}{}_{\rho'\mu'''} \gamma^\mu \Sigma^{\nu'\rho'} \gamma^{\mu'''} = 0,\tag{166}$$

$$\gamma^\mu \Sigma^{\nu'\rho'} \gamma^{\sigma'} \gamma^{\mu'''} \hat{R}^{\mu'\nu}{}_{\nu'\mu''} \hat{R}^{\mu''\rho}{}_{\rho'\mu'''} \hat{R}^{\mu'''\sigma}{}_{\sigma'\mu''''} \eta_{\mu\mu'} =$$

$$= q^2 \hat{R}^{\nu\sigma'}_{\sigma''\nu'} \hat{R}^{\rho\sigma}_{\sigma'\rho'} \gamma^{\sigma''} \Sigma^{\nu'\rho'}, \quad (167)$$

$$\begin{aligned} \gamma^\mu \gamma^{\nu'} \Sigma^{\rho'\sigma'} \gamma^{\mu''''} \hat{R}^{\mu'\nu}_{\nu'\mu''} \hat{R}^{\mu''\rho}_{\rho'\mu'''} \hat{R}^{\mu'''\sigma}_{\sigma'\mu''''} \eta_{\mu\mu'} &= \\ &= q^2 \hat{R}^{\nu'\sigma}_{\sigma'\nu''} \hat{R}^{\nu\rho}_{\rho'\nu'} \Sigma^{\rho'\sigma'} \gamma^{\nu''}. \end{aligned} \quad (168)$$

$$\begin{aligned} \gamma^\mu \gamma^{\rho'_1} \gamma^{\rho'_2} \gamma^{\rho'_3} \gamma^{\rho'_4} \gamma^\nu \hat{R}^{\mu'\rho_1}_{\rho'_1\mu''} \hat{R}^{\mu''\rho_2}_{\rho'_2\mu'''} \hat{R}^{\mu'''\rho_3}_{\rho'_3\mu''''} \hat{R}^{\mu''''\rho_4}_{\rho'_4\mu''} \eta_{\mu\mu'} &= \\ &= q^2 \gamma^{\rho'_4} \gamma^{\rho'_1} \gamma^{\rho'_2} \gamma^{\rho'_3} \hat{R}^{\rho_1\rho'_4}_{\rho'_4\rho'_1} \hat{R}^{\rho_2\rho'_4}_{\rho'_4\rho'_2} \hat{R}^{\rho_3\rho_4}_{\rho'_4\rho'_3} \\ &+ q^4 \gamma^{\rho'_2} \gamma^{\rho'_3} \gamma^{\rho'_4} \gamma^{\rho'_1} \hat{R}^{\rho_1\rho_2}_{\rho'_2\rho'_1} \hat{R}^{\rho_1\rho_3}_{\rho'_3\rho'_1} \hat{R}^{\rho'_1\rho_4}_{\rho'_4\rho'_1} = \\ &= q^2 \gamma^{\rho'_3} \gamma^{\rho'_2} \gamma^{\rho'_1} \gamma^{\rho_4} \hat{R}^{\rho_2\rho'_3}_{\rho'_3\rho'_2} \hat{R}^{\rho_1\rho_2}_{\rho'_2\rho'_1} \hat{R}^{\rho_1\rho_3}_{\rho'_3\rho'_1} \\ &+ q^4 \gamma^{\rho_1} \gamma^{\rho'_4} \gamma^{\rho'_3} \gamma^{\rho'_2} \hat{R}^{\rho_2\rho'_4}_{\rho'_4\rho'_2} \hat{R}^{\rho_2\rho'_3}_{\rho'_3\rho'_2} \hat{R}^{\rho_3\rho_4}_{\rho'_4\rho'_3}. \end{aligned} \quad (169)$$

Substituting  $\hat{R}$  and  $q$  by  $\hat{R}^{-1}$  and  $q^{-1}$ , respectively, leads to further relations. To get the undeformed counterparts of (166)-(169) one simply has to replace the  $\hat{R}$ -matrices by a normal twist, i.e.

$$\hat{R}^{\mu\nu}_{\rho\sigma} = \delta^\mu_\sigma \delta^\nu_\rho. \quad (170)$$

The corresponding relations for 'inverse'  $\gamma$ - and  $\Sigma$ -matrices follow from the formulae in (166)-(169) via the replacements (157) together with

$$\Sigma^{\mu\nu} \rightarrow \Sigma^{-1}_{\mu\nu}, \quad (P_A)^{\mu\nu}_{\rho\sigma} \rightarrow (P_A)^{\rho\sigma}_{\mu\nu}. \quad (171)$$

By the way this correspondence also applies to the formulae in (161)-(165).

Remarkably, products of  $\Sigma$ -matrices with  $\gamma^5$  reduce to linear combinations of  $\Sigma$ -matrices:

$$\begin{aligned} \Sigma^{+3} \gamma^5 &= -2q \lambda_+^{-1} \Sigma^{+0} + \lambda \lambda_+^{-1} \Sigma^{+3}, \\ \Sigma^{+0} \gamma^5 &= -2q^{-1} \lambda_+^{-1} \Sigma^{+3} - \lambda \lambda_+^{-1} \Sigma^{+0}, \\ \Sigma^{+-} \gamma^5 &= -2\lambda_+^{-1} \Sigma^{30} + \lambda \lambda_+^{-1} \Sigma^{+-}, \\ \Sigma^{30} \gamma^5 &= -2\lambda_+^{-1} \Sigma^{+-} - \lambda \lambda_+^{-1} \Sigma^{30}, \\ \Sigma^{3-} \gamma^5 &= 2q \lambda_+^{-1} \Sigma^{0-} - \lambda \lambda_+^{-1} \Sigma^{3-}, \\ \Sigma^{0-} \gamma^5 &= 2q^{-1} \lambda_+^{-1} \Sigma^{3-} + \lambda \lambda_+^{-1} \Sigma^{0-}. \end{aligned} \quad (172)$$

The corresponding formulae for 'inverse'  $\gamma$ - and  $\Sigma$ -matrices we get by the substitutions  $\Sigma^{\mu\nu} \rightarrow \Sigma^{-1}_{\mu\nu}$  together with  $\lambda \rightarrow -\lambda$ .

The matrices  $1$ ,  $\gamma^5$ ,  $\gamma^\mu$ ,  $\gamma^5 \gamma^\mu$ , and  $\Sigma^{\mu\nu}$  play an important role in physics, since they lead to bilinear forms with definite transformation properties (see

also the discussion in the succeeding section). On these grounds, the following expansion of an arbitrary matrix could prove useful:

$$\begin{aligned}
A &= a_0 \mathbb{1} + a_5 \gamma^5 + v_\mu \gamma^\mu + a_\mu \gamma^5 \gamma^\mu + T_{\mu\nu} \Sigma^{\mu\nu}, \\
a_0 &= -\frac{1}{2} \lambda_+^{-1} \text{Tr}_q(A), \\
a_5 &= -\frac{1}{2} \lambda_+^{-1} \text{Tr}_q(\gamma^5 A), \\
v_\mu &= \frac{1}{2} \eta_{\mu\nu} \text{Tr}_q(\gamma^\nu A), \\
a_\mu &= -\frac{1}{2} \eta_{\mu\nu} \text{Tr}_q(\gamma^5 \gamma^\nu A), \\
T_{\mu\nu} &= -\lambda_+^{-1} \eta_{\mu\mu'} \eta_{\nu\nu'} \text{Tr}_q(\Sigma^{\nu'\mu'} A).
\end{aligned} \tag{173}$$

There is also a version with 'inverse' matrices:

$$\begin{aligned}
A &= a^0 \mathbb{1} + a^5 \gamma_5^{-1} + v^\mu \gamma_\mu^{-1} + a^\mu \gamma_5^{-1} \gamma_\mu^{-1} + T^{\mu\nu} \Sigma_{\mu\nu}^{-1}, \\
a^0 &= -\frac{1}{2} \lambda_+^{-1} \text{Tr}_q(A), \\
a^5 &= -\frac{1}{2} \lambda_+^{-1} \text{Tr}_q(\gamma_5^{-1} A), \\
v^\mu &= \frac{1}{2} \eta^{\mu\nu} \text{Tr}_q(\gamma_\nu^{-1} A), \\
a^\mu &= -\frac{1}{2} \eta^{\mu\nu} \text{Tr}_q(\gamma_5^{-1} \gamma_\nu^{-1} A), \\
T^{\mu\nu} &= -\lambda_+^{-1} \eta^{\mu\mu'} \eta^{\nu\nu'} \text{Tr}_q(\Sigma_{\nu'\mu'}^{-1} A),
\end{aligned} \tag{174}$$

but now with the quantum trace from Eq. (119).

The expansion in (173) refers to the basis in (141). Additionally, we could also find an expansion for the basis in (108):

$$\begin{aligned}
A &= a_0 \mathbb{1} + a_{16} \Gamma^{+30-} + v_\mu \gamma^\mu + a_K \Gamma^K + T_{\mu\nu} \Sigma^{\mu\nu}, \\
a_0 &= -\frac{1}{2} \lambda_+^{-1} \text{Tr}_q(A), \\
a_{16} &= -\frac{1}{8} q^{27} [[2]]_{q^2}^{-7} [[3]]_{q^2}^{-4} \text{Tr}_q(\Gamma^{1234} A), \\
v_\mu &= \frac{1}{2} \eta_{\mu\nu} \text{Tr}_q(\gamma^\nu A), \\
a_K &= \text{Tr}_q(\Gamma_K A),
\end{aligned}$$

$$T_{\mu\nu} = -\lambda_+^{-1} \eta_{\mu\mu'} \eta_{\nu\nu'} \text{Tr}_q(\Sigma^{\nu'\mu'} A), \quad (175)$$

where  $K$  stands for an index running over  $K \in \{+30, +3-, +0-, 30-\}$ . The expansion with 'inverse' matrices now reads

$$\begin{aligned} A &= a^0 \mathbb{1} + a^{16} \Gamma_{+30-}^{-1} + v^\mu \gamma_\mu^{-1} + a^K \Gamma_K^{-1} + T^{\mu\nu} \Sigma_{\mu\nu}^{-1}, \\ a^0 &= -\frac{1}{2} \lambda_+^{-1} \text{Tr}_q(A), \\ a^{16} &= -\frac{1}{8} q^{27} [[2]]_{q^2}^{-7} [[3]]_{q^2}^{-4} \text{Tr}_q(\Gamma_{+30-}^{-1} A), \\ v^\mu &= \frac{1}{2} \eta^{\mu\nu} \text{Tr}_q(\gamma_\nu^{-1} A), \\ a^K &= \text{Tr}_q(\Gamma_K^{-1} A), \\ T^{\mu\nu} &= -\lambda_+^{-1} \eta^{\mu\mu'} \eta^{\nu\nu'} \text{Tr}_q(\Sigma_{\nu'\mu'}^{-1} A), \end{aligned} \quad (176)$$

where we again have to take the quantum trace from Eq. (119).

## 4 $q$ -Deformed Dirac spinors and bilinear covariants

Now, we should have everything together to introduce  $q$ -deformed Dirac spinors together with their bilinear covariants. It is also our aim to discuss the behavior of these objects under  $q$ -deformed Lorentz transformations (notice that the reasonings in this section refer to  $q$ -deformed Minkowski space). For this reason, we start our reasonings with a short review of the  $q$ -deformed Lorentz group [10, 11]. Finally, we should remark that in this section we work with Pauli matrices that are multiplied by an overall factor  $\lambda_+^{1/2}$  compared to the expressions given in part I. In doing so, we avoid additional factors in our formulae and the matrix  $\gamma^0$  takes on the very simple form

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (177)$$

#### 4.1 The quantum group $SL_q(2)$ and its coactions on the quantum plane

In view of their importance for what follows, we repeat some facts given already in part I. First, the algebra of  $SL_q(2)$  is generated by four non-commuting generators  $a, b, c$ , and  $d$ , subject to the relations

$$\begin{aligned} ab &= qba, & ac &= qca, & bc &= cb, & bd &= qdb, \\ cd &= qdc, & ad - da &= (q - q^{-1})bc. \end{aligned} \quad (178)$$

These generators can be recognized as entries of a matrix

$$M^i_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (179)$$

This notation enables us to write the relations in (178) with the  $\hat{R}$ -matrix of  $U_q(su(2))$  [58, 59]:

$$\hat{R}^{ij}_{kl} M^k_r M^l_s = M^i_k M^j_l \hat{R}^{kl}_{rs}, \quad (180)$$

and the Hopf structure of  $SL_q(2)$  takes on the form

$$\begin{aligned} \Delta(M^i_j) &= M^i_k \otimes M^k_j, \\ S(M^i_j) &= \varepsilon_{lj} M^l_k \varepsilon^{ik}, \\ S^{-1}(M^i_j) &= \varepsilon_{jl} M^l_k \varepsilon^{ki}, \\ \epsilon(M^i_j) &= \delta^i_j, \end{aligned} \quad (181)$$

where the non-zero entries of the  $q$ -deformed spinor metric are given by

$$\varepsilon_{12} = -q^{-1/2}, \quad \varepsilon_{21} = q^{1/2}. \quad (182)$$

The quantum group  $SL_q(2)$  gives rise to coactions on the so-called quantum plane. (If the reader is not familiar with the notion of a coaction we recommend to consult Refs. [36–38]). Remember that the coordinates on the quantum plane are viewed as  $q$ -analogs of spinors. The commutation relations for symmetrized spinors read as

$$x^1 x^2 = q x^2 x^1 \quad \text{and} \quad x_1 x_2 = q^{-1} x_2 x_1. \quad (183)$$

(There is also an antisymmetrized version of the quantum plane [52, 60]. However, our reasonings in this section do not depend on whether we work



with symmetrized or antisymmetrized spinors.) Notice that indices of spinors are raised and lowered by the spinor metric, i.e., for example,  $x_i = \varepsilon_{ij}x^j$ . Finally, let us recall the left- and right-coactions on covariant as well as contravariant spinor coordinates [see also part I]:

$$\begin{aligned}\beta_L(x^i) &= M^i_j \otimes x^j, \\ \beta_L(x_i) &= S^{-1}(M^j_i) \otimes x_j,\end{aligned}\tag{184}$$

$$\begin{aligned}\beta_R(x^i) &= x^j \otimes (M^T)^i_j, \\ \beta_R(x_i) &= x_j \otimes S((M^T)^j_i).\end{aligned}\tag{185}$$

## 4.2 The quantum group $SL_q(2, \mathbb{C})$

Next, we would like to describe the quantum group  $SL_q(2, \mathbb{C})$ , as it is necessary for the discussion of the transformation properties of Dirac spinors and their bilinear covariants. In part I we clarified the structure of the quantum groups  $SL_q(2)$  and  $SU_q(2)$ . The quantum groups  $SU_q(2)$  and  $SL_q(2, \mathbb{C})$  obey the same algebra relations and Hopf structure as  $SL_q(2)$ , but differ in their  $*$ -structure. In the case of  $SU_q(2)$ , conjugation maps the quantum group  $SL_q(2)$  onto itself. In the case of  $SL_q(2, \mathbb{C})$ , however, the conjugated generators of  $SL_q(2)$  live in a second copy of  $SL_q(2)$ .

For this to become more clear, we denote the generators of the first copy of  $SL_q(2)$  by  $M^i_j$  and those of the second copy by  $\tilde{M}^i_j$ . The two copies of  $SL_q(2)$  can be combined to form one algebra if we assume for the two sets of generators that

$$\hat{R}^{ij}_{kl} M^k_s \tilde{M}^l_t = \tilde{M}^i_m M^j_n \hat{R}^{mn}_{st},\tag{186}$$

where  $\hat{R}^{ij}_{kl}$  again stands for the  $\hat{R}$ -matrix of  $U_q(su(2))$ .

If we work with left-coactions of the two copies of  $SL_q(2)$ , their generators  $M^i_j$  and  $\tilde{M}^i_j$  are related to each other by

$$\overline{M^i_j} \equiv \bar{M}^i_j = S^{-1}(\tilde{M}^j_i) \quad \Rightarrow \quad (M^\dagger)^i_j = S^{-1}(\tilde{M}^i_j),\tag{187}$$

where  $(M^T)^\alpha_\beta = M^\beta_\alpha$  and  $M^\dagger = \overline{M^T} = \bar{M}^T$ . However, this  $*$ -structure is not consistent with right-coactions of the two copies of  $SL_q(2)$ , since it would violate the  $*$ -coalgebra axiom

$$\beta_R(\overline{x^i}) = \overline{\beta_R(x^i)}.\tag{188}$$

If we deal with right-coactions, we instead have to chose the  $*$ -structure

$$\bar{M}^i_j = S(\tilde{M}^j_i) \quad \Rightarrow \quad (M^\dagger)^i_j = S(\tilde{M}^i_j). \quad (189)$$

Finally, it should be mentioned that the two copies of  $SL_q(2)$  play symmetrical roles. With the help of the  $*$ -Hopf algebra axiom  $* \circ S^{-1} = S \circ *$  we can invert the identities in (187) and (189). This way, we respectively arrive at

$$\tilde{\bar{M}}^i_j = S^{-1}(M^j_i) \quad \text{and} \quad \tilde{\bar{M}}^i_j = S(M^j_i). \quad (190)$$

### 4.3 Relationship between $q$ -Lorentz group and $SL_q(2, \mathbb{C})$

In part I we introduced the generators of the quantum Lorentz group in spinor basis, i.e.

$$\Lambda^{\alpha\dot{\beta}}_{\gamma\dot{\delta}} = M^\alpha_\gamma \otimes \tilde{M}^{\dot{\beta}}_{\dot{\delta}} \in SL_q(2) \otimes SL_q(2), \quad \alpha, \dot{\beta}, \gamma, \dot{\delta} \in \{1, 2\}. \quad (191)$$

It arises the question how to define the generators of quantum Lorentz group in a vector basis, i.e.  $\Lambda^\mu_\nu$ ,  $\mu, \nu \in \{+, 3, 0, -\}$ ? To answer this question we consider the coaction [cf. part I]

$$\begin{aligned} \beta_L(X^\mu) &= (\sigma^\mu)_{\alpha\dot{\beta}} \beta_L(X^{\alpha\dot{\beta}}) \\ &= (\sigma^\mu)_{\alpha\dot{\beta}} M^\alpha_\gamma \tilde{M}^{\dot{\beta}}_{\dot{\delta}} \otimes X^{\gamma\dot{\delta}} \\ &= (\sigma^\mu)_{\alpha\dot{\beta}} M^\alpha_\gamma \tilde{M}^{\dot{\beta}}_{\dot{\delta}} (\sigma_\nu^{-1})^{\gamma\dot{\delta}} \otimes X^\nu \\ &\stackrel{!}{=} \Lambda^\mu_\nu \otimes X^\nu. \end{aligned} \quad (192)$$

From the last equality of the above calculation we read off that

$$\Lambda^\mu_\nu = (\sigma^\mu)_{\alpha\dot{\beta}} M^\alpha_\gamma \tilde{M}^{\dot{\beta}}_{\dot{\delta}} (\sigma_\nu^{-1})^{\gamma\dot{\delta}}. \quad (193)$$

Similar arguments lead us to

$$\Lambda^\mu_\nu = (\bar{\sigma}^\mu)_{\dot{\alpha}\beta} (\tilde{M})^{\dot{\alpha}}_{\dot{\gamma}} M^\beta_\delta (\bar{\sigma}_\nu^{-1})^{\dot{\gamma}\delta}. \quad (194)$$

The generators in (193) as well as (194) indeed fulfill the crucial equation

$$\eta_{\mu\nu} \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} = \eta_{\mu'\nu'}. \quad (195)$$

To prove this relation we first transform the quantum metric and the group generators into a spinor basis. Exploiting the properties of the Pauli matrices

together with those of the generators  $M^\alpha_\gamma$  and  $\tilde{M}^{\dot{\beta}}_{\dot{\delta}}$  then should show us the validity of the above identity.

The relations in (193) and (194) can also be written in a slightly different form. To this end, let us return to the coaction  $\beta_L(X^\mu)$ :

$$\begin{aligned}\beta_L(X^\mu) &= \beta_L(X^\alpha_{\dot{\alpha}}(\sigma^\mu)_{\alpha}^{\dot{\alpha}}) \\ &= (\sigma^\mu)_{\alpha}^{\dot{\alpha}} M^\alpha_{\beta} S^{-1}(\tilde{M}^{\dot{\beta}}_{\dot{\alpha}}) \otimes X^\beta_{\dot{\beta}} \\ &= (\sigma^\mu)_{\alpha}^{\dot{\alpha}} M^\alpha_{\beta} (M^\dagger)^{\dot{\beta}}_{\dot{\alpha}} \otimes X^\beta_{\dot{\beta}} \\ &\stackrel{!}{=} \Lambda^\mu_{\nu} \otimes X^\nu = \Lambda^\mu_{\nu} (\sigma^\nu)_{\beta}^{\dot{\beta}} \otimes X^\beta_{\dot{\beta}}.\end{aligned}\tag{196}$$

The last calculation should tell us that

$$(\Lambda_L)^\mu_{\nu} (\sigma^\nu)_{\beta}^{\dot{\beta}} = M^\alpha_{\beta} (\sigma^\mu)_{\alpha}^{\dot{\alpha}} \bar{M}^{\dot{\alpha}}_{\dot{\beta}},\tag{197}$$

or, for short,

$$(\Lambda_L)^\mu_{\nu} \sigma^\nu = M^T \sigma^\mu \bar{M}.\tag{198}$$

Similar arguments yield the following list of matrix identities:

$$\begin{aligned}(\Lambda_L)^\mu_{\nu} \sigma^\nu &= M_L^T \sigma^\mu \bar{M}_L, & (\Lambda_L)^\mu_{\nu} \bar{\sigma}^\nu &= \tilde{M}_L^T \bar{\sigma}^\mu \tilde{M}_L, \\ (\Lambda_R^T)^\mu_{\nu} \sigma^\nu &= M_R \sigma^\mu M_R^\dagger, & (\Lambda_R^T)^\mu_{\nu} \bar{\sigma}^\nu &= \tilde{M}_R \bar{\sigma}^\mu \tilde{M}_R^\dagger,\end{aligned}\tag{199}$$

$$\begin{aligned}(\Lambda_L)^\mu_{\nu} \sigma_\mu^{-1} &= \tilde{M}_L \sigma_\nu^{-1} \tilde{M}_L^T, & (\Lambda_L)^\mu_{\nu} \bar{\sigma}_\mu^{-1} &= \bar{M}_L \bar{\sigma}_\nu^{-1} M_L^T, \\ (\Lambda_R^T)^\mu_{\nu} \sigma_\mu^{-1} &= \tilde{M}_R^\dagger \sigma_\nu^{-1} \tilde{M}_R, & (\Lambda_R^T)^\mu_{\nu} \bar{\sigma}_\mu^{-1} &= M_R^\dagger \bar{\sigma}_\nu^{-1} M_R.\end{aligned}\tag{200}$$

Notice that the two labels  $L$  and  $R$  were introduced to indicate the type of coaction (left- or right-coaction) to which each of the above relations corresponds to. Furthermore, one should realize that the derivation of the identities in (200) starts from applying coactions to the defining relations of 'inverse' Pauli matrices, i.e.

$$X^\mu (\sigma_\mu^{-1})_{\alpha}^{\dot{\beta}} = X_{\alpha}^{\dot{\beta}}, \quad X^\mu (\bar{\sigma}_\mu^{-1})_{\dot{\alpha}}^{\beta} = X_{\dot{\alpha}}^{\beta}.\tag{201}$$

## 4.4 Definition of Dirac spinors and bilinear covariants

We introduce the *Dirac spinor* as

$$\Psi_a \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\lambda}_{\dot{\alpha}} \end{pmatrix},\tag{202}$$

i.e. it is a 4-spinor written with the two 2-spinors

$$\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \end{pmatrix}. \quad (203)$$

Then the *Hermitian conjugate* of a Dirac spinor should become

$$(\Psi^\dagger)^a \equiv (\bar{\psi}^{\dot{\alpha}}, \lambda^\alpha), \quad (204)$$

where

$$\bar{\psi}^{\dot{\alpha}} = (\bar{\psi}^{\dot{1}}, \bar{\psi}^{\dot{2}}) = -\varepsilon^{\dot{\alpha}\dot{\alpha}'} \bar{\psi}_{\dot{\alpha}'}, \quad \lambda^\alpha = (\lambda^1, \lambda^2) = \varepsilon^{\alpha\alpha'} \lambda_{\alpha'}. \quad (205)$$

In physics, however, it is more convenient to work with the so-called *Dirac conjugated 4-spinor*:

$$\bar{\Psi}^a \equiv (\lambda^\alpha, \bar{\psi}^{\dot{\alpha}}) = (\Psi^\dagger) \gamma^0. \quad (206)$$

Next, we would like to introduce covariant bilinear forms. In doing so, we need a second Dirac spinor

$$\Phi_a \equiv \begin{pmatrix} \chi_\alpha \\ \bar{\phi}_{\dot{\alpha}} \end{pmatrix}. \quad (207)$$

Its Hermitian and Dirac conjugate are defined in complete analogy to those for the Dirac spinor  $\Psi$ . With the two Dirac spinors  $\Psi$  and  $\Phi$  at hand we are now in a position to introduce the bilinear covariants

$$\begin{aligned} \bar{\Phi} \Psi &= \phi^\beta \psi_\beta + \bar{\chi}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}}, \\ \bar{\Phi} \gamma^5 \Psi &= -\phi^\beta \psi_\beta + \bar{\chi}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}}, \\ \bar{\Phi} \gamma^\mu \Psi &= \phi^\alpha (\sigma^\mu)_\alpha{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}} + \bar{\chi}^{\dot{\alpha}} (\bar{\sigma}^\mu)_{\dot{\alpha}}{}^\beta \psi_\beta, \\ \bar{\Phi} \gamma^\mu \gamma^5 \Psi &= \phi^\alpha (\sigma^\mu)_\alpha{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}} - \bar{\chi}^{\dot{\alpha}} (\bar{\sigma}^\mu)_{\dot{\alpha}}{}^\beta \psi_\beta, \\ \bar{\Phi} \Sigma^{\mu\nu} \Psi &= \phi^\alpha (\sigma^{\mu\nu})_\alpha{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}} + \bar{\chi}^{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}}, \\ \bar{\Phi} \Sigma^{\mu\nu} \gamma^5 \Psi &= -\phi^\alpha (\sigma^{\mu\nu})_\alpha{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}} + \bar{\chi}^{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}}. \end{aligned} \quad (208)$$

Notice, that the first five covariants are sufficient to constitute a basis, since  $\Sigma^{\mu\nu} \gamma^5$  can be expressed as a linear combination of  $\Sigma$ -matrices [cf. the relations in (172)]. The expressions in terms of 2-spinors are a direct consequence of the definition of Dirac spinors and the explicit form of the matrices  $\gamma^5$ ,  $\gamma^\mu$ , and  $\Sigma^{\mu\nu}$ .

## 4.5 Conjugation properties

Before we discuss the transformation properties of the bilinear forms in (208) we would like to say a few words about their behavior under conjugation. To begin with, we list the conjugation properties of the different types of 2-spinors:

$$\overline{\psi_\beta} = \bar{\psi}^{\dot{\beta}}, \quad \overline{\bar{\psi}^{\dot{\beta}}} = \psi_\beta, \quad \overline{\lambda^\beta} = \bar{\lambda}_{\dot{\beta}}, \quad \overline{\bar{\lambda}_{\dot{\beta}}} = \lambda^\beta. \quad (209)$$

It should be clear that this conjugation assignment is consistent with (204).

Next, we come to the conjugation properties of the Pauli matrices, which can be derived most easily from their defining relations, i.e.

$$X^\mu = (\sigma^\mu)_{\alpha\dot{\beta}} X^{\alpha\dot{\beta}} = (\bar{\sigma}^\mu)_{\dot{\alpha}\beta} X^{\dot{\alpha}\beta}. \quad (210)$$

Conjugating the expressions in (210) and exploiting the identities

$$\overline{X^\mu} = (-1)^{\delta_{0\mu}} X_\mu, \quad \overline{X^{\alpha\dot{\beta}}} = X_{\beta\dot{\alpha}}, \quad (211)$$

we finally get

$$\begin{aligned} \overline{(\sigma^\mu)_{\alpha\dot{\beta}}} &= (\sigma^\mu)_{\dot{\beta}\alpha}^\dagger = (-1)^{\delta_{0\mu}} (\sigma_\mu)^{\beta\dot{\alpha}}, \\ \overline{(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}} &= (\bar{\sigma}^\mu)_{\beta\dot{\alpha}}^\dagger = (-1)^{\delta_{0\mu}} (\bar{\sigma}_\mu)^{\dot{\beta}\alpha}. \end{aligned} \quad (212)$$

Starting from the defining relations for 'inverse' Pauli matrices and repeating similar arguments as above one should be able to show that

$$\begin{aligned} \overline{(\sigma_\mu^{-1})^{\alpha\dot{\beta}}} &= (\sigma_\mu^{-1})^{\dot{\beta}\alpha} = (-1)^{\delta_{0\mu}} (\sigma^{-1\mu})_{\beta\dot{\alpha}}, \\ \overline{(\bar{\sigma}_\mu^{-1})^{\dot{\alpha}\beta}} &= (\bar{\sigma}_\mu^{-1})^{\beta\dot{\alpha}} = (-1)^{\delta_{0\mu}} (\bar{\sigma}^{-1\mu})_{\dot{\beta}\alpha}. \end{aligned} \quad (213)$$

Next, we turn to the conjugation properties of two-dimensional spin matrices. To this end, we consider their definitions [cf. part I]

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &= (P_A)^{\mu\nu}{}_{\kappa\lambda} (\sigma^\kappa \bar{\sigma}^\lambda)_\alpha{}^\beta, \\ (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} &= (P_A)^{\mu\nu}{}_{\kappa\lambda} (\bar{\sigma}^\kappa \sigma^\lambda)_{\dot{\alpha}}{}^{\dot{\beta}}, \end{aligned} \quad (214)$$

$$\begin{aligned} (\sigma_{\mu\nu}^{-1})_\alpha{}^\beta &= (P_A)^{\kappa\lambda}{}_{\mu\nu} (\sigma_\kappa^{-1} \bar{\sigma}_\lambda^{-1})_\alpha{}^\beta, \\ (\bar{\sigma}_{\mu\nu}^{-1})_{\dot{\alpha}}{}^{\dot{\beta}} &= (P_A)^{\kappa\lambda}{}_{\mu\nu} (\bar{\sigma}_\kappa^{-1} \sigma_\lambda^{-1})_{\dot{\alpha}}{}^{\dot{\beta}}. \end{aligned} \quad (215)$$

We conjugate both sides of each equality in (215). Using the identities in (212) and (213) together with

$$(P_A)^{\mu\nu}{}_{\kappa\lambda} = (P_A)^{\kappa\lambda}{}_{\mu\nu}, \quad (216)$$

we should finally arrive at

$$\begin{aligned} \overline{(\sigma^{\mu\nu})_{\alpha\beta}} &= (\sigma^{\mu\nu})_{\dot{\beta}\dot{\alpha}}^\dagger = (-1)^{\delta_{0\mu}+\delta_{0\nu}} (\bar{\sigma}_{\nu\mu})^{\dot{\beta}\dot{\alpha}}, \\ \overline{(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}} &= (\bar{\sigma}^{\mu\nu})_{\beta\alpha}^\dagger = (-1)^{\delta_{0\mu}+\delta_{0\nu}} (\sigma_{\nu\mu})^{\beta\alpha}, \end{aligned} \quad (217)$$

$$\begin{aligned} \overline{(\sigma_{\mu\nu}^{-1})^{\alpha\beta}} &= (\sigma_{\mu\nu}^{-1})_{\dot{\beta}\dot{\alpha}}^\dagger = (-1)^{\delta_{0\mu}+\delta_{0\nu}} (\bar{\sigma}^{-1\ \nu\mu})_{\dot{\beta}\dot{\alpha}}, \\ \overline{(\bar{\sigma}_{\mu\nu}^{-1})^{\dot{\alpha}\dot{\beta}}} &= (\bar{\sigma}_{\mu\nu}^{-1})_{\beta\alpha}^\dagger = (-1)^{\delta_{0\mu}+\delta_{0\nu}} (\sigma^{-1\ \nu\mu})_{\beta\alpha}. \end{aligned} \quad (218)$$

Now, we are ready to give the conjugation properties of the bilinear co-variants:

$$\begin{aligned} \overline{\bar{\Phi}\Psi} &= \bar{\Psi}\Phi, \\ \overline{\bar{\Phi}\gamma^5\Psi} &= -\bar{\Psi}\gamma^5\Phi, \\ \overline{\bar{\Phi}\gamma^\mu\Psi} &= (-1)^{\delta_{0\mu}} \bar{\Psi}\gamma_\mu\Phi, \\ \overline{\bar{\Phi}\gamma^\mu\gamma^5\Psi} &= (-1)^{\delta_{0\mu}} \bar{\Psi}\gamma_\mu\gamma^5\Phi, \\ \overline{\bar{\Phi}\Sigma^{\mu\nu}\Psi} &= (-1)^{\delta_{0\mu}+\delta_{0\nu}} \bar{\Psi}\Sigma_{\nu\mu}\Phi, \\ \overline{\bar{\Phi}\Sigma^{\mu\nu}\gamma^5\Psi} &= (-1)^{\delta_{0\mu}+\delta_{0\nu}+1} \bar{\Psi}\Sigma_{\nu\mu}\gamma^5\Phi, \end{aligned} \quad (219)$$

where

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\nu, \quad \Sigma_{\mu\nu} = \eta_{\mu\mu'}\eta_{\nu\nu'}\Sigma^{\mu'\nu'}. \quad (220)$$

The identities in (219) are readily checked by means of the definitions in (208) together with the results in (209), (212), and (218). An example shall illustrate this:

$$\begin{aligned} \overline{\bar{\Phi}\gamma^\mu\Psi} &= \overline{\phi^\alpha(\sigma^\mu)_\alpha{}^{\dot{\beta}}\bar{\lambda}_{\dot{\beta}}} + \overline{\bar{\chi}^{\dot{\alpha}}(\bar{\sigma}^\mu)_{\dot{\alpha}}{}^\beta\psi_\beta} \\ &= \lambda^\beta \overline{(\sigma^\mu)_\alpha{}^{\dot{\beta}}\bar{\phi}_{\dot{\alpha}}} + \bar{\psi}^{\dot{\beta}} \overline{(\bar{\sigma}^\mu)_{\dot{\alpha}}{}^\beta\chi_{\dot{\alpha}}} \\ &= (-1)^{\delta_{\mu 0}} \lambda^\beta (\sigma_\mu)_{\beta}{}^{\dot{\alpha}} \bar{\phi}_{\dot{\alpha}} + (-1)^{\delta_{\mu 0}} \bar{\psi}^{\dot{\beta}} (\bar{\sigma}_\mu)_{\dot{\beta}}{}^{\dot{\alpha}} \chi_{\dot{\alpha}} \\ &= (-1)^{\delta_{0\mu}} \bar{\Psi}\gamma_\mu\Phi. \end{aligned} \quad (221)$$

In analogy to the undeformed case the second relation in (219) should be consistent with the identities

$$(\gamma^5)^\dagger = \gamma^5 \quad \text{and} \quad \gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5, \quad (222)$$

which are a direct consequence of (102) and (103). Indeed we have

$$\begin{aligned} \overline{\Phi} \gamma^5 \Psi &= \overline{\Phi^\dagger \gamma^0 \gamma^5 \Psi} = \Psi^\dagger (\gamma^5)^\dagger (\gamma^0)^\dagger \Phi \\ &= \Psi^\dagger \gamma^0 \gamma^0 (\gamma^5)^\dagger \gamma^0 \Phi = -\Psi^\dagger \gamma^0 \gamma^5 \Phi \\ &= -\bar{\Psi} \gamma^5 \Phi. \end{aligned} \quad (223)$$

Moreover, inspection of the explicit form of the  $\gamma$ -matrices shows that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = (-1)^{\delta_{\mu 0}}\gamma_\mu, \quad (224)$$

which, in turn, implies

$$\begin{aligned} \gamma^0(\gamma^\mu \gamma^5)^\dagger\gamma^0 &= \gamma^0 \gamma^5 (\gamma^\mu)^\dagger \gamma^0 \\ &= -\gamma^5 \gamma^0 (\gamma^\mu)^\dagger \gamma^0 = (-1)^{\delta_{\mu 0}+1} \gamma^5 \gamma_\mu \\ &= (-1)^{\delta_{\mu 0}} \gamma_\mu \gamma^5. \end{aligned} \quad (225)$$

Once again, (224) and (225) are in agreement with (219). In a similar way the last two identities in (219) require for the  $\Sigma$ -matrices that

$$\begin{aligned} \gamma^0(\Sigma^{\mu\nu})^\dagger\gamma^0 &= (-1)^{\delta_{\mu 0}+\delta_{\nu 0}} \Sigma_{\nu\mu}, \\ \gamma^0(\Sigma^{\mu\nu} \gamma^5)^\dagger\gamma^0 &= (-1)^{\delta_{\mu 0}+\delta_{\nu 0}+1} \Sigma_{\nu\mu} \gamma^5. \end{aligned} \quad (226)$$

## 4.6 Transformation properties of Dirac spinors

Next, we discuss the behavior of Dirac spinors under  $q$ -deformed Lorentz transformations. Each Dirac spinor is made up of two 2-spinors, so its transformation properties are completely determined by the coactions of  $q$ -deformed Lorentz group on its 2-spinors:

$$\begin{aligned} \beta_L(\lambda^\alpha) &= M^\alpha{}_\beta \otimes \lambda^\beta, \\ \beta_L(\psi_\alpha) &= S^{-1}((M^T)^\alpha{}_\beta) \otimes \psi_\beta, \\ \beta_R(\lambda^\alpha) &= \lambda^\beta \otimes M^\beta{}_\alpha, \end{aligned} \quad (227)$$

$$\beta_R(\psi_\alpha) = \psi_\beta \otimes S((M^T)^\beta{}_\alpha), \quad (228)$$

and

$$\begin{aligned} \beta_L(\bar{\lambda}^{\dot{\alpha}}) &= \tilde{M}^{\dot{\alpha}}{}_{\dot{\beta}} \otimes \bar{\lambda}^{\dot{\beta}}, \\ \beta_L(\bar{\lambda}_{\dot{\alpha}}) &= S^{-1}((\tilde{M}^T)^{\dot{\alpha}}{}_{\dot{\beta}}) \otimes \bar{\lambda}_{\dot{\beta}}, \end{aligned} \quad (229)$$

$$\begin{aligned} \beta_R(\bar{\lambda}^{\dot{\alpha}}) &= \bar{\lambda}^{\dot{\beta}} \otimes \tilde{M}^{\dot{\beta}}{}_{\dot{\alpha}}, \\ \beta_R(\bar{\lambda}_{\dot{\alpha}}) &= \bar{\lambda}_{\dot{\beta}} \otimes S((\tilde{M}^T)^{\dot{\beta}}{}_{\dot{\alpha}}). \end{aligned} \quad (230)$$

With these formulae at hand the left-coaction of the  $q$ -deformed Lorentz group on a Dirac spinor becomes

$$\beta_L(\Psi) = \beta_L \begin{pmatrix} \psi_\alpha \\ \bar{\lambda}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} S^{-1}(M^T)^\alpha{}_\beta \otimes \psi_\beta \\ S^{-1}(\tilde{M}^T)^{\dot{\alpha}}{}_{\dot{\beta}} \otimes \bar{\lambda}_{\dot{\beta}} \end{pmatrix}. \quad (231)$$

Notice that  $M$  and  $\tilde{M}$  denote the matrices to the two copies of  $SL_q(2)$ , i.e.

$$M^\alpha{}_\beta \otimes \tilde{M}^{\dot{\alpha}}{}_{\dot{\beta}} \in SL_q(2, \mathbb{C}) \simeq SL_q(2) \otimes SL_q(2). \quad (232)$$

Further, we work with the transposed matrices  $M^T$  and  $\tilde{M}^T$ . This enables us to think in terms of matrix multiplication, as is indicated through the position of spinor indices (in our convention the first index of a matrix labels its rows and the second one its columns).

Now, we want to get rid of the generators with a tilde. We achieve this by the  $*$ -structures of  $SL_q(2, \mathbb{C})$  that are given through the identifications in (187) and (189). From the  $*$ -structure being relevant for left-coactions [cf. Eq. (187)] we find

$$S^{-1}(\tilde{M}^T)^{\dot{\alpha}}{}_{\dot{\beta}} = \overline{(M^T)}^{\dot{\beta}}{}_{\dot{\alpha}} = (M^\dagger)^{\dot{\alpha}}{}_{\dot{\beta}}, \quad (233)$$

or, for short,

$$S^{-1}(\tilde{M}^T) = \bar{M}. \quad (234)$$

We can also avoid the antipodes in (231). From a comparison of

$$\varepsilon_{\alpha\beta} M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} = \varepsilon_{\alpha'\beta'} \quad (235)$$

with the identities

$$S^{-1}(M^{\beta'}{}_\alpha) M^\beta{}_{\beta'} = S(M^\beta{}_{\beta'}) M^{\beta'}{}_\alpha = \delta^\beta{}_\alpha, \quad (236)$$



we find

$$\begin{aligned} S^{-1}(M^\beta{}_\alpha) &= \varepsilon_{\alpha\alpha'} M^{\alpha'}{}_{\beta'} \varepsilon^{\beta'\beta} = -(\varepsilon^T)^{\beta\beta'} (M^T)^{\beta'}{}_{\alpha'} (\varepsilon^T)^{\alpha'\alpha}, \\ S(M^\beta{}_\alpha) &= \varepsilon^{\beta\beta'} M^{\alpha'}{}_{\beta'} \varepsilon_{\alpha'\alpha} = -\varepsilon^{\beta\beta'} (M^T)^{\beta'}{}_{\alpha'} \varepsilon^{\alpha'\alpha}, \end{aligned} \quad (237)$$

where we have used  $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta}$ . From now on matrix multiplication between  $\varepsilon$  and  $M$  will always be understood as defined in (237) with two adjacent indices. On these grounds we write the results in (237) as

$$S^{-1}(M^T) = -\varepsilon M \varepsilon, \quad S(M) = -\varepsilon M^T \varepsilon. \quad (238)$$

Before proceeding any further let us sum up that for left coactions the relationship between the generators of the two tensor factors of  $SL_q(2, \mathbb{C})$  is given by

$$\tilde{M} = S(M^\dagger) = -\varepsilon^T \bar{M} \varepsilon^T, \quad (239)$$

whereas for right coactions we have

$$\tilde{M} = S^{-1}(M^\dagger) = -\varepsilon \bar{M} \varepsilon. \quad (240)$$

With (234) and (238) the left-coaction of a Dirac spinor finally becomes

$$\begin{aligned} \beta_L(\Psi) &= \begin{pmatrix} -\varepsilon^{\alpha\alpha''} M^{\alpha''}{}_{\alpha'''} \varepsilon^{\alpha'''\alpha'} & 0 \\ 0 & \bar{M}^{\dot{\alpha}}{}_{\dot{\alpha}'} \end{pmatrix} \otimes \begin{pmatrix} \psi_{\alpha'} \\ \bar{\lambda}_{\dot{\alpha}'} \end{pmatrix}, \\ &= \begin{pmatrix} -\varepsilon M \varepsilon & 0 \\ 0 & \bar{M} \end{pmatrix} \otimes \begin{pmatrix} \psi \\ \bar{\lambda} \end{pmatrix} \\ &\equiv L_S(\Lambda) \otimes \Psi, \end{aligned} \quad (241)$$

where we introduced the Lorentz matrix  $L_S$  for notational convenience. Of course, these ideas carry over to right coactions of Dirac spinors. With the help of (228) and (230) we get, at once,

$$\begin{aligned} \beta_R(\Psi) &= \beta_R \begin{pmatrix} \psi_\alpha \\ \bar{\lambda}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_{\alpha'} \otimes S(M^T)^{\alpha'}{}_\alpha \\ \bar{\lambda}_{\dot{\alpha}'} \otimes S(\tilde{M}^T)^{\dot{\alpha}'}{}_{\dot{\alpha}} \end{pmatrix} \\ &= \begin{pmatrix} 1 \otimes S(M)^{\alpha}{}_{\alpha'} & 0 \\ 0 & 1 \otimes S(\tilde{M})^{\dot{\alpha}}{}_{\dot{\alpha}'} \end{pmatrix} \begin{pmatrix} \psi_{\alpha'} \otimes 1 \\ \bar{\lambda}_{\dot{\alpha}'} \otimes 1 \end{pmatrix}. \end{aligned} \quad (242)$$

Making use of (189) and (238) the last expression can be rewritten as

$$\beta_R(\Psi) = \begin{pmatrix} -1 \otimes \varepsilon^{\alpha\alpha'''} (M^T)^{\alpha'''}{}_{\alpha''} \varepsilon^{\alpha''\alpha'} & 0 \\ 0 & 1 \otimes (M^\dagger)^{\dot{\alpha}}{}_{\dot{\alpha}'} \end{pmatrix} \begin{pmatrix} \psi_{\alpha'} \otimes 1 \\ \bar{\lambda}_{\dot{\alpha}'} \otimes 1 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} -1 \otimes \varepsilon M^T \varepsilon & 0 \\ 0 & 1 \otimes M^\dagger \end{pmatrix} \begin{pmatrix} \psi \otimes 1 \\ \bar{\lambda} \otimes 1 \end{pmatrix} \\
&\equiv (1 \otimes R_S(\Lambda))(\Psi \otimes 1),
\end{aligned} \tag{243}$$

where the Lorentz matrix  $R_S$  is defined in an obvious way.

It is not very difficult to repeat the above reasonings for the Hermitian conjugate as well as the Dirac conjugate. Thus, we left the details to the reader and restrict ourselves to stating the results, only. Instead, we give a summary of all coactions on the different types of 4-spinors. We write our results in an index-free notation. From what we have done so far it should be rather clear how to write them in a form with all indices being displayed.

We begin with left coactions on Dirac spinors. In doing so, we need the matrices

$$L_S(\Lambda) = \begin{pmatrix} -\varepsilon M \varepsilon & 0 \\ 0 & \bar{M} \end{pmatrix}, \quad L_S^\dagger(\Lambda) = \begin{pmatrix} -\varepsilon^T M^\dagger \varepsilon^T & 0 \\ 0 & M^T \end{pmatrix}. \tag{244}$$

With these matrices the left coactions of  $q$ -deformed Lorentz group on 4-spinors take on the rather intuitive form

$$\begin{aligned}
\beta_L(\Psi^a) &= L_S(\Lambda)^a_b \otimes \Psi^b, \\
\beta_L(\Psi_a^\dagger) &= (1 \otimes \Psi_b^\dagger)(L_S^\dagger(\Lambda)^b_a \otimes 1), \\
\beta_L(\bar{\Psi}_a) &= (1 \otimes \bar{\Psi}_b)((\gamma^0 L_S^\dagger(\Lambda) \gamma^0)^b_a \otimes 1).
\end{aligned} \tag{245}$$

In dealing with right coactions we need the matrices

$$R_S(\Lambda) = \begin{pmatrix} -\varepsilon M^T \varepsilon & 0 \\ 0 & M^\dagger \end{pmatrix}, \quad R_S^\dagger(\Lambda) = \begin{pmatrix} -\varepsilon^T \bar{M} \varepsilon^T & 0 \\ 0 & M \end{pmatrix}. \tag{246}$$

In analogy to (245) we have

$$\begin{aligned}
\beta_R(\Psi^a) &= (1 \otimes R_S(\Lambda)^a_b)(\Psi^b \otimes 1), \\
\beta_R(\Psi_a^\dagger) &= \Psi_b^\dagger \otimes R_S^\dagger(\Lambda)^b_a, \\
\beta_R(\bar{\Psi}_a) &= \bar{\Psi}_b \otimes (\gamma^0 R_S^\dagger(\Lambda) \gamma^0)^b_a.
\end{aligned} \tag{247}$$

The reader may have realized that we are working with a kind of Weyl representation. A characteristic feature of the Weyl representation is that 4-spinors are written as two 2-spinors that transform independently from each

other under Lorentz transformations. A short look at (227)-(230) tells us that the transformation properties of a two-spinor are completely determined by the type of its index (dotted or undotted, covariant or contravariant). On these grounds, it should be obvious that the 4-spinor

$$\Psi^c \equiv \begin{pmatrix} \lambda_\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \begin{pmatrix} \lambda^\beta \\ \bar{\psi}^{\dot{\beta}} \end{pmatrix} = C\bar{\Psi}^T \quad (248)$$

behaves under  $q$ -deformed Lorentz transformations in very much the same way as  $\Psi$ . It can be viewed as the 4-spinor one obtains by charge conjugation. In analogy to the undeformed case the Majorana condition

$$\Psi^c = \Psi, \quad (249)$$

gives  $\lambda = \psi$ , i.e.

$$\Psi^c = \Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix}. \quad (250)$$

## 4.7 Transformation properties of bilinear covariants

Now, we are well prepared to discuss the transformation properties of the bilinear covariants. We can assume that in the expressions of (208) the transformation properties are assigned to the spinors, i.e. we simply have to transform the 4-spinors according to (245) and (247), while the  $\gamma$ -matrices remain unchanged.

First of all, we consider the transformation of  $\bar{\Phi}\Psi$ :

$$\begin{aligned} \beta_L(\bar{\Phi}\Psi) &= \beta_L(\bar{\Phi})\beta_L(\Psi) \\ &= (1 \otimes \bar{\Phi})(\gamma^0 L_S^\dagger(\Lambda) \gamma^0 L_S(\Lambda) \otimes \Psi) \\ &= \mathbb{1} \otimes \bar{\Phi}\Psi. \end{aligned} \quad (251)$$

First, we apply the homomorphism property of the coaction. Then we insert the expressions from (245). The last equality follows from the calculation

$$\begin{aligned} (\gamma^0 L_S^\dagger(\Lambda) \gamma^0) L_S(\Lambda) &= \begin{pmatrix} M^T & 0 \\ 0 & -\varepsilon^T M^\dagger \varepsilon^T \end{pmatrix} \begin{pmatrix} -\varepsilon M \varepsilon & 0 \\ 0 & \bar{M} \end{pmatrix} \\ &= \begin{pmatrix} M^T S^{-1}(M^T) & 0 \\ 0 & S(\bar{M})\bar{M} \end{pmatrix} = \mathbb{1}, \end{aligned} \quad (252)$$

The second step uses (238) and the third one is a direct consequence of the Hopf algebra axioms

$$M_{(2)} S^{-1}(M)_{(1)} = M_{(1)} S(M)_{(2)} = 1. \quad (253)$$

Likewise, we can show that

$$\gamma^0 R_S^\dagger(\Lambda) \gamma^0 R_S(\Lambda) = \mathbb{1}, \quad (254)$$

from which we conclude

$$\beta_R(\bar{\Phi}\Psi) = \bar{\Phi}\Psi \otimes \mathbb{1}. \quad (255)$$

To sum up,  $\bar{\Phi}\Psi$  transforms like a scalar under the coactions of  $q$ -deformed Lorentz group. Finally, a short look at (252) and (254) tells us that  $L_S(\Lambda)$  and  $R_S(\Lambda)$  are not unitary matrices.

Next, let us examine how the bicovariants with  $\gamma$ -matrices behave under Lorentz transformations:

$$\begin{aligned} \beta_L(\bar{\Phi} \gamma^\mu \Psi) &= \beta_L(\bar{\Phi}) \gamma^\mu \beta_L(\Psi) \\ &= (\mathbb{1} \otimes \bar{\Phi})(\gamma^0 L_S^\dagger(\Lambda) \gamma^0 \gamma^\mu L_S(\Lambda) \otimes \Psi) \\ &= \Lambda^\mu{}_\nu \otimes \bar{\Phi} \gamma^\nu \Psi. \end{aligned} \quad (256)$$

The last equality results from the identity

$$(\Lambda_L)^\mu{}_\nu \gamma^\nu = (\gamma^0 L_S^\dagger(\Lambda) \gamma^0) \gamma^\mu L_S(\Lambda), \quad (257)$$

which can be recognized as a four-dimensional generalization of (199). Its proof reads as follows:

$$\begin{aligned} (\gamma^0 L_S^\dagger(\Lambda) \gamma^0) \gamma^\mu L_S(\Lambda) &= \begin{pmatrix} M^T & 0 \\ 0 & S(\bar{M}) \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} S^{-1}(M^T) & 0 \\ 0 & \bar{M} \end{pmatrix} \\ &= \begin{pmatrix} 0 & M^T \sigma^\mu \bar{M} \\ S(\bar{M}) \bar{\sigma}^\mu S^{-1}(M^T) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & M^T \sigma^\mu \bar{M} \\ \tilde{M}^T \bar{\sigma}^\mu \tilde{M} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\Lambda_L)^\mu{}_\nu \sigma^\nu \\ (\Lambda_L)^\mu{}_\nu \bar{\sigma}^\nu & 0 \end{pmatrix} \\ &= (\Lambda_L)^\mu{}_\nu \gamma^\nu. \end{aligned} \quad (258)$$

Similar arguments lead us to

$$(\Lambda_R^T)^\mu{}_\nu \gamma^\nu = (\gamma^0 R_S^\dagger(\Lambda) \gamma^0) \gamma^\mu R_S(\Lambda), \quad (259)$$

which, in turn, means

$$\beta_R(\bar{\Phi} \gamma^\mu \Psi) = (\Lambda^T)^\mu{}_\nu \otimes \bar{\Phi} \gamma^\nu \Psi. \quad (260)$$

In summary,  $\bar{\Phi} \gamma^\mu \Psi$  transforms under  $q$ -deformed Lorentz transformations as a 4-vector.

As already mentioned, (257) and (259) can be seen as four-dimensional generalizations of (199). There are also four-dimensional generalizations of (200):

$$\begin{aligned} (\Lambda_L)^\mu{}_\nu \gamma_\mu^{-1} &= L_S(\Lambda) \gamma_\nu^{-1} (\gamma_0^{-1} L_S^\dagger(\Lambda) \gamma_0^{-1}), \\ (\Lambda_R^T)^\mu{}_\nu \gamma_\mu^{-1} &= R_S(\Lambda) \gamma_\nu^{-1} (\gamma_0^{-1} R_S^\dagger(\Lambda) \gamma_0^{-1}). \end{aligned} \quad (261)$$

To prove these relations one can proceed in the following manner:

$$\begin{aligned} \beta_L(X^\mu \gamma_\mu^{-1}) &= (\Lambda_L)^\mu{}_\nu \otimes X^\nu \gamma_\mu^{-1} = \begin{pmatrix} 0 & \beta_L(X^\mu \sigma_\mu^{-1}) \\ \beta_L(X^\mu \bar{\sigma}_\mu^{-1}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \tilde{M} \sigma_\nu^{-1} \tilde{M}^T \\ \bar{M} \bar{\sigma}_\nu^{-1} M^T & 0 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{M} & 0 \\ 0 & \bar{M} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\nu^{-1} \\ \bar{\sigma}_\nu^{-1} & 0 \end{pmatrix} \begin{pmatrix} M^T & 0 \\ 0 & \tilde{M}^T \end{pmatrix} \\ &= L_S(\Lambda) \gamma_\nu^{-1} (\gamma_0^{-1} L_S^\dagger(\Lambda) \gamma_0^{-1}). \end{aligned} \quad (262)$$

The covariants with  $\Sigma$ -matrices transform like antisymmetric tensors under  $q$ -deformed Lorentz transformations:

$$\begin{aligned} \beta_L(\bar{\Phi} \Sigma^{\mu\nu} \Psi) &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} (\bar{\Phi} \Sigma^{\mu'\nu'} \Psi), \\ \beta_R(\bar{\Phi} \Sigma^{\mu\nu} \Psi) &= (\Lambda^T)^\mu{}_{\mu'} (\Lambda^T)^\nu{}_{\nu'} (\bar{\Phi} \Sigma^{\mu'\nu'} \Psi). \end{aligned} \quad (263)$$

We illustrate the ideas leading to these relations by the following calculation:

$$\begin{aligned} \beta_L(\bar{\Phi} \Sigma^{\mu\nu} \Psi) &= (P_A)^{\mu\nu}{}_{\mu'\nu'} \beta_L(\bar{\Phi} \gamma^\mu \gamma^\nu \Psi) \\ &= (P_A)^{\mu\nu}{}_{\mu'\nu'} \Lambda^{\mu'}{}_{\mu''} \Lambda^{\nu'}{}_{\nu''} \otimes (\bar{\Phi} \gamma^{\mu''} \gamma^{\nu''} \Psi) \end{aligned}$$

$$\begin{aligned}
&= \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} (P_A)^{\mu'\nu'}_{\mu''\nu''} \otimes (\bar{\Phi} \gamma^{\mu''} \gamma^{\nu''} \Psi) \\
&= \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \otimes (\bar{\Phi} \Sigma^{\mu'\nu'} \Psi).
\end{aligned} \tag{264}$$

The first equality is the definition of the  $\Sigma$ -matrices. The second equality can be seen as an application of (256). To understand the third identity one has to realize that the generators of  $q$ -deformed Lorentz group obey the relations

$$\hat{R}^{\mu\nu}_{\mu'\nu'} \Lambda^{\mu'}_{\mu''} \Lambda^{\nu'}_{\nu''} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \hat{R}^{\mu'\nu'}_{\mu''\nu''}, \tag{265}$$

where the  $\hat{R}$ -matrix refers to  $q$ -deformed Lorentz group. It should be clear that these relations remain valid if the  $\hat{R}$ -matrix is substituted by any of its projectors.

To describe the transformation properties of the covariants with  $\gamma^5$  we need the definition of the  $q$ -determinant for the quantum Lorentz group [61, 62]. In our notation it is determined by

$$(\det_q \Lambda) \varepsilon_{\mu'\nu'\rho'\sigma'} = \varepsilon_{\mu\nu\rho\sigma} \Lambda^\sigma_{\sigma'} \Lambda^\rho_{\rho'} \Lambda^\nu_{\nu'} \Lambda^\mu_{\mu'}. \tag{266}$$

Contracting both sides of this relation with  $\varepsilon^{\sigma'\rho'\nu'\mu'}$  yields

$$\det_q \Lambda = -\frac{1}{2} q^8 [[2]]_{q^2}^{-2} [[3]]_{q^2}^{-1} \varepsilon_{\mu\nu\rho\sigma} \Lambda^\sigma_{\sigma'} \Lambda^\rho_{\rho'} \Lambda^\nu_{\nu'} \Lambda^\mu_{\mu'} \varepsilon^{\sigma'\rho'\nu'\mu'}. \tag{267}$$

Using (266) we find

$$\begin{aligned}
\beta_L(\bar{\Phi} \gamma^5 \Psi) &= \frac{q^5}{[[2]]_{q^2} [[3]]_{q^2}} \varepsilon_{\mu\nu\rho\sigma} \beta_L(\bar{\Phi} \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu \Psi) \\
&= \frac{q^5}{[[2]]_{q^2} [[3]]_{q^2}} \varepsilon_{\mu\nu\rho\sigma} \Lambda^\sigma_{\sigma'} \Lambda^\rho_{\rho'} \Lambda^\nu_{\nu'} \Lambda^\mu_{\mu'} \otimes (\bar{\Phi} \gamma^{\sigma'} \gamma^{\rho'} \gamma^{\nu'} \gamma^{\mu'} \Psi) \\
&= \frac{q^5}{[[2]]_{q^2} [[3]]_{q^2}} (\det_q \Lambda) \varepsilon_{\mu'\nu'\rho'\sigma'} \otimes (\bar{\Phi} \gamma^{\sigma'} \gamma^{\rho'} \gamma^{\nu'} \gamma^{\mu'} \Psi) \\
&= \det_q \Lambda \otimes \bar{\Phi} \gamma^5 \Psi,
\end{aligned} \tag{268}$$

and, likewise,

$$\beta_R(\bar{\Phi} \gamma^5 \Psi) = \bar{\Phi} \gamma^5 \Psi \otimes \det_q \Lambda^T$$

In this manner, we conclude that  $\bar{\Phi} \gamma^5 \Psi$  transforms as a pseudoscalar. With this observation it is not difficult to verify that  $\bar{\Phi} \gamma^\mu \gamma^5 \Psi$  behaves as a pseudo-vector, i.e.

$$\begin{aligned}
\beta_L(\bar{\Phi} \gamma^\mu \gamma^5 \Psi) &= (\det_q \Lambda) \Lambda^\mu_{\nu} \otimes \bar{\Phi} \gamma^\nu \gamma^5 \Psi, \\
\beta_R(\bar{\Phi} \gamma^\mu \gamma^5 \Psi) &= \bar{\Phi} \gamma^\nu \gamma^5 \Psi \otimes (\det_q \Lambda^T) (\Lambda^T)^\mu_{\nu}.
\end{aligned} \tag{269}$$

## 5 Conclusion

Let us end with some comments on what we have done so far. In part I and II of the present article we gave a systematic treatment of  $q$ -deformed spinor calculus. In part I attention was concentrated on Weyl spinors, while part II dealt with the foundations of the Dirac formalism. One of the highlights of part I was the presentation of the Weyl spinor Fierz identities, which play a very important role in supersymmetry, especially in the derivation of transformation properties of superfields. In part II we could finally discuss Dirac spinors and their bilinear covariants along the same line of reasonings as in the undeformed case.

Furthermore, we could derive a rather complete list of  $q$ -analogs of trace relations and rearrangement formulae concerning Pauli, Dirac, and spin matrices. Our results show striking similarities to their undeformed counterparts. The reason for this observation lies in the fact that  $q$ -deformation does not change the representation theoretic content of the theory. On these grounds, our list of relations could prove very useful for calculations in quantum field theory.

However, it should become clear that  $q$ -deformation leads to a more sophisticated structure. There are often several  $q$ -deformed variants to one and the same undeformed object. Thus, combining  $q$ -deformed objects in the right way often requires a lot of testing and experience accompanied by a deep understanding of the subject.

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## A Some explicit formulae for the four-dimensional Euclidean space

In Sec. 3.1.1 we gave two basis of the Clifford algebra of four-dimensional  $q$ -deformed Euclidean space. In terms of the generators of the Clifford algebra

the elements of the two bases become

$$\begin{aligned}
\gamma^{[1]\gamma^i]_q} &= \gamma^1 \gamma^i, & i = 2, 3, \\
\gamma^{[1]\gamma^4]_q} &= \gamma^1 \gamma^4 - q^{-1} \lambda_+^{-1} \mathbb{1}, \\
\gamma^{[2]\gamma^3]_q} &= \gamma^2 \gamma^3 - \lambda_+^{-1} \mathbb{1}, \\
\gamma^{[i]\gamma^4]_q} &= \gamma^2 \gamma^4, & i = 2, 3,
\end{aligned} \tag{270}$$

$$\begin{aligned}
\gamma^{[1]\gamma^2\gamma^3]_q} &= \gamma^1 \gamma^2 \gamma^3 - \frac{1}{2} q^{-1} \gamma^1, \\
\gamma^{[1]\gamma^i\gamma^4]_q} &= \gamma^1 \gamma^i \gamma^4 + \frac{1}{2} q^{-1} \gamma^i, & i = 2, 3, \\
\gamma^{[2]\gamma^3\gamma^4]_q} &= \gamma^2 \gamma^3 \gamma^4 - \frac{1}{2} q^{-1} \gamma^4,
\end{aligned} \tag{271}$$

$$\begin{aligned}
\gamma^5 \gamma^1 &= -\gamma^1 + 2q \gamma^1 \gamma^2 \gamma^3, \\
\gamma^5 \gamma^2 &= -\gamma^2 - 2q \gamma^1 \gamma^2 \gamma^3, \\
\gamma^5 \gamma^3 &= \gamma^3 + 2q \gamma^1 \gamma^2 \gamma^3, \\
\gamma^5 \gamma^4 &= \gamma^4 - 2q \gamma^1 \gamma^2 \gamma^3,
\end{aligned} \tag{272}$$

$$\begin{aligned}
\gamma^5 &= -2\gamma^1 \gamma^4 - 2q \gamma^2 \gamma^3 + 2q^2 \lambda_+ \gamma^1 \gamma^2 \gamma^3 \gamma^4 + \mathbb{1} \\
\gamma^{[1]\gamma^2\gamma^3\gamma^4]_q} &= 4[[2]]_{q^2}^4 [[3]]_{q^2}^2 \gamma^1 \gamma^2 \gamma^3 \gamma^4 - 4[[2]]_{q^2}^3 [[3]]_{q^2}^2 (\gamma^2 \gamma^3 + q^{-1} \gamma^1 \gamma^4) \\
&\quad + 2q^{-1} [[2]]_{q^2}^3 [[3]]_{q^2}^2 \mathbb{1}.
\end{aligned} \tag{273}$$

The matrices representing the elements of the two bases are

$$\begin{aligned}
\Gamma^{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{13} &= \begin{pmatrix} 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma^{14} &= \lambda_+^{-1} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}, & \Gamma^{23} &= \lambda_+^{-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\Gamma^{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\end{aligned}$$



$$\begin{aligned}
\Gamma^{123} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1/2} & 0 \\ 0 & 0 & 0 & 0 \\ -q^{-5/2} & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{124} &= -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-3/2} \\ q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma^{134} &= -\frac{1}{2} \begin{pmatrix} 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q^{-3/2} & 0 & 0 \end{pmatrix}, & \Gamma^{234} &= \frac{1}{2} q^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma^{51} &= q^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{52} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1/2} \\ q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma^{54} &= q^{1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{53} &= - \begin{pmatrix} 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -q^{-1/2} & 0 & 0 \end{pmatrix}, \\
\Gamma^{1234} &= -2q^{-1} [[2]]_{q^2}^3 [[3]]_{q^2}^2 \gamma^5.
\end{aligned}$$

We would like to write down the  $\Sigma$ -matrices. Clearly, we have  $\Sigma^{\mu\nu} = \Gamma^{\mu\nu}$  for  $(\mu\nu) \in \{(12), (13), (14), (23), (24), (34)\}$  [cf. (47) and (82)]. Moreover, it holds  $\Sigma^{\mu\mu} = 0$  for all  $\mu = 1, \dots, 4$ . The remaining  $\Sigma$ -matrices are

$$\begin{aligned}
\Sigma^{21} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Sigma^{31} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Sigma^{32} &= \lambda_+^{-1} \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}, & \Sigma^{41} &= \lambda_+^{-1} \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \\
\Sigma^{42} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Sigma^{43} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q & 0 \end{pmatrix}. \tag{274}
\end{aligned}$$

## B Some explicit expressions for Minkowski space

In terms of the generators of the Clifford algebra the elements of the two bases in (108) and (141) become

$$\begin{aligned}
\gamma^{[+\gamma^3]_q} &= \gamma^+ \gamma^3, \\
\gamma^{[+\gamma^0]_q} &= \gamma^+ \gamma^0, \\
\gamma^{[+\gamma^-]_q} &= \gamma^+ \gamma^- - q \lambda_+^{-1} \mathbb{1}, \\
\gamma^{[3\gamma^0]_q} &= \gamma^3 \gamma^0, \\
\gamma^{[3\gamma^-]_q} &= \gamma^3 \gamma^-, \\
\gamma^{[0\gamma^-]_q} &= \gamma^0 \gamma^-,
\end{aligned} \tag{275}$$

$$\begin{aligned}
\gamma^{[+\gamma^3\gamma^0]_q} &= \gamma^+ \gamma^3 \gamma^0 + \frac{q}{2} \lambda \lambda_+^{-1} \gamma^+, \\
\gamma^{[+\gamma^3\gamma^-]_q} &= \gamma^+ \gamma^3 \gamma^- + q \lambda_+^{-1} \gamma^3 + \frac{q^2}{2} \lambda \lambda_+^{-1} \gamma^0, \\
\gamma^{[+\gamma^0\gamma^-]_q} &= \gamma^+ \gamma^0 \gamma^- + \frac{1}{2} \lambda \lambda_+^{-1} \gamma^3 + \frac{1}{2} (q^4 + 1) q^{-1} \lambda_+^{-1} \gamma^0, \\
\gamma^{[3\gamma^0\gamma^-]_q} &= \gamma^3 \gamma^0 \gamma^- + \frac{q}{2} \lambda \lambda_+^{-1} \gamma^{-1},
\end{aligned} \tag{276}$$

$$\begin{aligned}
\gamma^5 \gamma^+ &= -2q^{-1} \gamma^+ \gamma^3 \gamma^0 - \lambda \lambda_+^{-1} \gamma^+, \\
\gamma^5 \gamma^3 &= 2\gamma^+ \gamma^0 \gamma^- - 2q^{-1} \lambda \gamma^+ \gamma^3 \gamma^- \\
&\quad + 2q \lambda_+^{-1} \gamma^0 - \lambda \lambda_+^{-1} \gamma^3, \\
\gamma^5 \gamma^0 &= 2q^{-2} \gamma^+ \gamma^3 \gamma^- + 2q^{-1} \lambda_+^{-1} \gamma^3 + \lambda \lambda_+^{-1} \gamma^0, \\
\gamma^5 \gamma^- &= 2q^{-1} \gamma^3 \gamma^0 \gamma^- + \lambda \lambda_+^{-1} \gamma^-,
\end{aligned} \tag{277}$$

$$\begin{aligned}
\gamma^5 &= -2q^{-1} \gamma^+ \gamma^- + 2q^{-1} \gamma^3 \gamma^0 - 2q^{-2} \lambda_+ \gamma^+ \gamma^3 \gamma^0 \gamma^- + \lambda \lambda_+^{-1} \mathbb{1} \\
\gamma^{[+\gamma^3\gamma^0\gamma^-]_q} &= -4q^{-14} [[2]]_q^3 [[3]]_q^2 \gamma^3 \gamma^0 + 4q^{-14} \lambda [[2]]_q^3 [[3]]_q^2 \gamma^+ \gamma^- \\
&\quad + 4q^{-16} [[2]]_q^4 [[3]]_q^2 \gamma^+ \gamma^3 \gamma^0 \gamma^- - 2q^{-14} \lambda [[2]]_q^2 [[3]]_q^2 \mathbb{1}.
\end{aligned} \tag{278}$$

Similar expressions exist for the two bases (117) and (145). Most of them follow from the expressions in (275)-(278) by applying the substitutions  $\gamma^\mu \rightarrow$

$\gamma_\mu^{-1}$ ,  $\mu \in \{+, 3, 0, -\}$ , with the exception of the following ones:

$$\begin{aligned}
\gamma_{[+}\gamma_3^{-1}\gamma_{0]q}^{-1} &= \gamma_+^{-1}\gamma_3^{-1}\gamma_0^{-1} - \frac{q}{2}\lambda\lambda_+^{-1}\gamma_+^{-1}, \\
\gamma_{[+}\gamma_3^{-1}\gamma_{-]q}^{-1} &= \gamma_+^{-1}\gamma_3^{-1}\gamma_-^{-1} + q\lambda_+^{-1}\gamma_3^{-1} - \frac{q^2}{2}\lambda\lambda_+^{-1}\gamma_0^{-1}, \\
\gamma_{[+}\gamma_0^{-1}\gamma_{-]q}^{-1} &= \gamma_+^{-1}\gamma_0^{-1}\gamma_-^{-1} - \frac{1}{2}\lambda\lambda_+^{-1}\gamma_3^{-1} + \frac{1}{2}(q^4 + 1)q^{-1}\lambda_+^{-1}\gamma_0^{-1}, \\
\gamma_{[3}\gamma_0^{-1}\gamma_{-]q}^{-1} &= \gamma_3^{-1}\gamma_0^{-1}\gamma_-^{-1} - \frac{q}{2}\lambda\lambda_+^{-1}\gamma_-^{-1},
\end{aligned} \tag{279}$$

$$\begin{aligned}
\gamma_5^{-1}\gamma_+^{-1} &= 2q^{-1}\gamma_+^{-1}\gamma_3^{-1}\gamma_0^{-1} - \lambda\lambda_+^{-1}\gamma_+^{-1}, \\
\gamma_5^{-1}\gamma_3^{-1} &= 2\gamma_+^{-1}\gamma_0^{-1}\gamma_-^{-1} - 2q^{-1}\lambda\gamma_+^{-1}\gamma_3^{-1}\gamma_-^{-1} \\
&\quad - 2q\lambda_+^{-1}\gamma_0^{-1} - \lambda\lambda_+^{-1}\gamma_3^{-1}, \\
\gamma_5^{-1}\gamma_0^{-1} &= -2q^{-2}\gamma_+^{-1}\gamma_3^{-1}\gamma_-^{-1} - 2q^{-1}\lambda_+^{-1}\gamma_3^{-1} + \lambda\lambda_+^{-1}\gamma_0^{-1}, \\
\gamma_5^{-1}\gamma_-^{-1} &= -2q^{-1}\gamma_3^{-1}\gamma_0^{-1}\gamma_-^{-1} + \lambda\lambda_+^{-1}\gamma_-^{-1},
\end{aligned} \tag{280}$$

$$\begin{aligned}
\gamma_5^{-1} &= -2q^{-1}\gamma_+^{-1}\gamma_-^{-1} - 2q^{-1}\gamma_3^{-1}\gamma_0^{-1} \\
&\quad + 2q^{-2}\lambda_+\gamma_+^{-1}\gamma_3^{-1}\gamma_0^{-1}\gamma_-^{-1} + \lambda\lambda_+^{-1}\mathbb{1} \\
\gamma_{[+}\gamma_3^{-1}\gamma_0^{-1}\gamma_{-]q}^{-1} &= 4q^{-14}[[2]]_q^3[[3]]_q^2\gamma_3^{-1}\gamma_0^{-1} + 4q^{-14}\lambda[[2]]_q^3[[3]]_q^2\gamma_+^{-1}\gamma_-^{-1} \\
&\quad - 4q^{-16}[[2]]_q^4[[3]]_q^2\gamma_+^{-1}\gamma_3^{-1}\gamma_0^{-1}\gamma_-^{-1} \\
&\quad - 2q^{-14}\lambda[[2]]_q^2[[3]]_q^2\mathbb{1}.
\end{aligned} \tag{281}$$

We give the matrices representing the elements of the two bases in (108) and (141):

$$\begin{aligned}
\Gamma^{+3} &= -\lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{3/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^{3/2} & 0 \end{pmatrix}, \quad \Gamma^{+0} = \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -q^{3/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \end{pmatrix}, \\
\Gamma^{+-} &= \lambda_+^{-1} \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \quad \Gamma^{30} = \lambda_+^{-1} \begin{pmatrix} -q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^{-2} \end{pmatrix}, \\
\Gamma^{3-} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{0-} = \lambda_+^{-1/2} \begin{pmatrix} 0 & -q^{-3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\Gamma^{+30} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{5/2} & 0 \\ 0 & 0 & 0 & 0 \\ -q^{1/2} & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{+3-} &= \frac{1}{2} q^2 \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
\Gamma^{+0-} &= \frac{1}{2} \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^2 \\ q^2 & 0 & 0 & 0 \\ 0 & q\lambda - q^{-2} & 0 & 0 \end{pmatrix}, & \Gamma^{30-} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -q^{3/2} \\ 0 & 0 & 0 & 0 \\ 0 & -q^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma^{5+} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & q^{3/2} & 0 \\ 0 & 0 & 0 & 0 \\ q^{-1/2} & 0 & 0 & 0 \end{pmatrix}, & \Gamma^{53} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -q^{-2} & 0 & 0 \end{pmatrix}, \\
\Gamma^{50} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \Gamma^{5-} &= \begin{pmatrix} 0 & 0 & 0 & -q^{1/2} \\ 0 & 0 & 0 & 0 \\ 0 & -q^{-3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma^{+30-} &= -2q^{-13} [[2]]_{q^2}^3 [[3]]_{q^2}^2 \gamma^5. \tag{282}
\end{aligned}$$

The matrix representations of the elements of the bases (117) and (145) are

$$\begin{aligned}
\Gamma_{+3}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma_{+0}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & q^{-3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^{1/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma_{+-}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, & \Gamma_{30}^{-1} &= \lambda_+^{-1} \begin{pmatrix} q^{-2} & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}, \\
\Gamma_{3-}^{-1} &= -\lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{3/2} & 0 & 0 & 0 \\ 0 & 0 & q^{3/2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma_{0-}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{3/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1/2} & 0 \end{pmatrix}, \\
\Gamma_{+30}^{-1} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & q^{-1/2} \\ 0 & 0 & 0 & 0 \\ 0 & q^{3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma_{+3-}^{-1} &= \frac{1}{2} q^2 \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\Gamma_{+0-}^{-1} &= \frac{1}{2} \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & q\lambda - q^{-2} & 0 \\ 0 & 0 & 0 & q^2 \\ -q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \Gamma_{3-0}^{-1} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & 0 \\ q^{5/2} & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma_{5+}^{-1} &= \begin{pmatrix} 0 & 0 & 0 & q^{-3/2} \\ 0 & 0 & 0 & 0 \\ 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma_{53}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -q^2 & 0 & 0 \end{pmatrix}, \\
\Gamma_{50}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \Gamma_{5-}^{-1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1/2} & 0 \\ 0 & 0 & 0 & 0 \\ -q^{3/2} & 0 & 0 & 0 \end{pmatrix}, \\
\Gamma_5^{-1} &= 2q^{-13} [[2]]_{q^2}^3 [[3]]_{q^2}^2 \gamma_5^{-1}. \tag{283}
\end{aligned}$$

The matrices  $\Sigma^{\mu\nu}$  that cannot be read off from the matrices for  $\Gamma^A$  are

$$\begin{aligned}
\Sigma^{3+} &= q^{-1/2} \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \Sigma^{0+} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{3/2} & 0 \end{pmatrix}, \\
\Sigma^{-+} &= \lambda_+^{-1} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}, & \Sigma^{33} &= \lambda \lambda_+^{-1} \begin{pmatrix} -q^1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^{-3} \end{pmatrix}, \\
\Sigma^{03} &= \lambda_+^{-1} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & -q^{-2} & 0 & 0 \\ 0 & 0 & -q^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \Sigma^{-3} &= -q^{-3/2} \lambda_+^{-1/2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Sigma^{-0} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-3/2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{284}
\end{aligned}$$

The matrices  $\Sigma_{\mu\nu}^{-1}$  that cannot be read off from the matrices for  $\Gamma_A^{-1}$  are

$$\begin{aligned}
\Sigma_{3+}^{-1} &= -q^{-3/2}\lambda_+^{-1/2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Sigma_{0+}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & -q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-3/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Sigma_{-+}^{-1} &= \lambda_+^{-1} \begin{pmatrix} -q^{-1} & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, & \Sigma_{33}^{-1} &= \lambda\lambda_+^{-1/2} \begin{pmatrix} q^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\Sigma_{03}^{-1} &= \lambda_+^{-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}, & \Sigma_{-3}^{-1} &= q^{-1/2}\lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\Sigma_{-0}^{-1} &= \lambda_+^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^{3/2} & 0 \end{pmatrix}.
\end{aligned} \tag{285}$$

## C $q$ -Deformed quantum spaces

The aim of this appendix is the following. For the quantum spaces under consideration we list the defining commutation relations. In addition to this, we write down the non-vanishing elements of their quantum metric and  $q$ -deformed epsilon tensor.

In the case of three-dimensional  $q$ -deformed Euclidean space the commutation relations between its coordinates  $X^A$ ,  $A \in \{+, 3, -\}$ , read

$$\begin{aligned}
X^3 X^\pm &= q^{\pm 2} X^\pm X^3, \\
X^- X^+ &= X^+ X^- + \lambda X^3 X^3.
\end{aligned} \tag{286}$$

The non-vanishing elements of the quantum metric are

$$g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}. \tag{287}$$

As usual, covariant coordinates are introduced by

$$X_A \equiv g_{AB} X^B, \tag{288}$$

with  $g_{AB}$  being the inverse of  $g^{AB}$ . The non-vanishing components of the three-dimensional  $q$ -deformed epsilon tensor take the form

$$\begin{aligned}\varepsilon^{-3+} &= -q^{-4}, & \varepsilon^{3-+} &= q^{-2}, \\ \varepsilon^{-+3} &= q^{-2}, & \varepsilon^{+-3} &= -q^{-2}, \\ \varepsilon^{3+-} &= -q^{-2}, & \varepsilon^{+3-} &= 1. \\ \varepsilon^{333} &= -q^{-2}\lambda.\end{aligned}\tag{289}$$

Next we come to four-dimensional  $q$ -deformed Euclidean space. For its coordinates  $X^i$ ,  $i = 1, \dots, 4$ , we have the relations

$$\begin{aligned}X^1 X^j &= q X^j X^1, \\ X^j X^4 &= q X^4 X^j, \quad j = 1, 2, \\ X^2 X^3 &= X^3 X^2, \\ X^4 X^1 &= X^1 X^4 + \lambda X^2 X^3.\end{aligned}\tag{290}$$

The metric has the non-vanishing components

$$g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q.\tag{291}$$

Its inverse denoted by  $g_{ij}$  can again be used to introduce covariant coordinates, i.e.

$$X_i \equiv g_{ij} X^j.\tag{292}$$

The non-vanishing components of the epsilon tensor of four-dimensional  $q$ -deformed Euclidean become (see also Refs. [43, 63–65])

$$\begin{aligned}\varepsilon^{1234} &= 1, & \varepsilon^{1432} &= -q^2, & \varepsilon^{2413} &= -q^2, \\ \varepsilon^{2134} &= -q, & \varepsilon^{4132} &= q^2, & \varepsilon^{4213} &= q^3, \\ \varepsilon^{1324} &= -1, & \varepsilon^{3412} &= q^2, & \varepsilon^{2341} &= -q^2, \\ \varepsilon^{3124} &= q, & \varepsilon^{4312} &= -q^3, & \varepsilon_{3241} &= q^2, \\ \varepsilon^{2314} &= q^2, & \varepsilon^{1243} &= -q, & \varepsilon^{2431} &= q^3, \\ \varepsilon^{3214} &= -q^2, & \varepsilon^{2143} &= q^2, & \varepsilon^{4231} &= -q^4, \\ \varepsilon^{1342} &= q, & \varepsilon^{1423} &= q^2, & \varepsilon^{3421} &= -q^3, \\ \varepsilon^{3142} &= -q^2, & \varepsilon^{4123} &= -q^2, & \varepsilon^{4321} &= q^4,\end{aligned}\tag{293}$$

together with the non-classical components

$$\varepsilon^{3232} = -\varepsilon^{2323} = -q^2\lambda.\tag{294}$$

Now, we come to  $q$ -deformed Minkowski space [12–14]. Its coordinates are subjected to the relations

$$\begin{aligned} X^\mu X^0 &= X^0 X^\mu, \quad \mu \in \{0, +, -, 3\}, \\ X^3 X^\pm - q^{\pm 2} X^\pm X^3 &= -q\lambda X^0 X^\pm, \\ X^- X^+ - X^+ X^- &= \lambda(X^3 X^3 - X^0 X^3), \end{aligned} \quad (295)$$

and its metric is given by

$$\eta^{00} = -1, \quad \eta^{33} = 1, \quad \eta^{+-} = -q, \quad \eta^{-+} = -q^{-1}. \quad (296)$$

As usual, the metric can be used to raise and lower indices. The non-vanishing components of the  $q$ -deformed epsilon tensor read

$$\begin{aligned} \varepsilon^{+30-} &= 1, & \varepsilon^{+-03} &= -q^{-2}, & \varepsilon^{3-+0} &= q^{-2}, \\ \varepsilon^{3+0-} &= -q^{-2}, & \varepsilon^{-+03} &= q^{-2}, & \varepsilon^{-3+0} &= q^{-4}, \\ \varepsilon^{+03-} &= -1, & \varepsilon^{0-+3} &= q^{-2}, & \varepsilon^{30-+} &= -q^{-2}, \\ \varepsilon^{0+3-} &= 1, & \varepsilon^{-0+3} &= -q^{-2}, & \varepsilon^{03-+} &= q^{-2}, \\ \varepsilon^{30+-} &= q^{-2}, & \varepsilon^{+3-0} &= -1, & \varepsilon^{3-0+} &= q^{-2}, \\ \varepsilon^{03+-} &= -q^{-2}, & \varepsilon^{3+-0} &= q^{-2}, & \varepsilon^{-30+} &= -q^{-4}, \\ \varepsilon^{+0-3} &= q^{-2}, & \varepsilon^{+-30} &= q^{-2}, & \varepsilon^{0-3+} &= -q^{-4}, \\ \varepsilon^{0+-3} &= -q^{-2}, & \varepsilon^{-+30} &= -q^{-2}, & \varepsilon^{-03+} &= q^{-4}, \end{aligned} \quad (297)$$

and

$$\begin{aligned} \varepsilon^{0-0+} &= q^{-3}\lambda, & \varepsilon^{-0+0} &= -q^{-3}\lambda, \\ \varepsilon^{0333} &= -q^{-2}\lambda, & \varepsilon^{3330} &= q^{-2}\lambda, \\ \varepsilon^{3033} &= +q^{-2}\lambda, & \varepsilon^{3030} &= -q^{-2}\lambda, \\ \varepsilon^{3303} &= -q^{-2}\lambda, & \varepsilon^{+0-0} &= -q^{-1}\lambda, \\ \varepsilon^{0303} &= q^{-2}\lambda, & \varepsilon^{0+0-} &= q^{-1}\lambda. \end{aligned} \quad (298)$$

Lowering the indices of the epsilon tensor is achieved by the quantum metric:

$$\varepsilon_{\mu\nu\rho\sigma} = \eta_{\mu\mu'}\eta_{\nu\nu'}\eta_{\rho\rho'}\eta_{\sigma\sigma'} \varepsilon^{\mu'\nu'\rho'\sigma'}. \quad (299)$$

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